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The method of numerical integration in exterior ballistics; ...

United States. Army. Ordnance Dept, Dunham ...

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THE METHOD OF NUMERICAL INTEGRATION IN EXTERIOR BALLISTICS

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WAR DEPARTMENT,
WASHINGTON, *October 29, 1919.*

The following publication, entitled "The Method of Numerical Integration in Exterior Ballistics," is published for the information and guidance of all concerned.

[062.13, A. G. O.]

BY ORDER OF THE SECRETARY OF WAR:

PEYTON C. MARCH,
General, Chief of Staff.

OFFICIAL:

P. C. HARRIS,
The Adjutant General.

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THE METHOD OF NUMERICAL INTEGRATION IN EXTERIOR BALLISTICS

By DUNHAM JACKSON, Ph. D.
Recently Captain, Ordnance Department, U. S. Army

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PREFACE.

The following pages represent an attempt to give in connected form the fundamental principles of the method of numerical integration in exterior ballistics, as it has been developed in this country during the last year and a half. The names of the men who have contributed most to its development, particularly Major Moulton and Professor Bliss, are mentioned at various places in the text, but a complete list of those who have had a share in it, and to whom the writer might appropriately make personal acknowledgment, would amount practically to an enumeration of all the officers, civilian investigators, and computers who have been connected with the work in ballistics in Washington and at the Aberdeen Proving Ground. Important advances in theory and practice are still made from day to day, and this pamphlet makes no pretense of being more than an introduction to the subject, even in its present state. As further sources of information, attention is called particularly to an article by Captain Bennett on "The Physical Bases of Ballistic Table Computations," presently to be published as an Ordnance Textbook, and the series of articles on ballistics appearing in current numbers of the Journal of the United States Artillery.

DUNHAM JACKSON,
Captain, Ordnance Department, U. S. Army.

WASHINGTON, D. C., *August 30, 1919.*

SUPPLEMENTARY NOTE.

At the time when the above words with regard to current progress were written, Dr. Gronwall was engaged in working out the practical details of a modification of Professor Bliss's method, by which the labor of computing differential variations is materially reduced. This improvement has been found so effective that a section describing it has been inserted in Chapter II. It is not to be inferred that progress has now come to an end, but, on the contrary, that further advances in one direction or another are to be looked for at any time.

D. J.

The UNIVERSITY OF MINNESOTA,
Minneapolis, Minn., November 29, 1919.

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THE METHOD OF NUMERICAL INTEGRATION IN EXTERIOR BALLISTICS.

INTRODUCTION.

THE LAWS OF AIR RESISTANCE.

1. *Scope of introductory remarks on air resistance.*—A projectile in flight is acted upon by two forces, its weight and the resistance of the air. For the high velocities with which the science of exterior ballistics is concerned, the latter force is comparable with the former, and may be many times the greater of the two. It is necessary, therefore, to know the laws of air resistance with considerable accuracy. A complete account of the experiments by which it has been attempted to determine these laws, and of the conclusions that have been drawn from the experiments, would fill a large volume. On the other hand, when a decision has once been reached as to the most probable form of the resistance functions, in the light of the best available knowledge, these functions can be reduced to tabular form, and the tables used in computation without further question as to their origin. Such tables are included in this pamphlet, and a very few words in the text give as much explanation of their meaning as is necessary for their use. It is the purpose of this introductory note to give something of a background for the bare statements in the text, without touching any but the most striking features.

2. *Dependence of the resistance on the velocity of the projectile.*—The primary observation is that for a given projectile the resistance of the air increases very rapidly with the speed of the motion. For velocities below and not too near the velocity of sound, and also for velocities considerably above that of sound, the resistance is approxlmately proportional to the square of the velocity, the constant of proportionality being considerably larger in the second case than in the first. In the intermediate zone, the change is much more rapid still. The existence of a critical region about the velocity of sound is to be accounted for by the fact that much of the energy of the projectile is dissipated in setting up wave motions in the air, and the general character of the disturbance is quite different according as the projectile moves slower or faster than the waves which it creates.

Before the introduction of the method of numerical integration, it was found most convenient to represent the resistance by a power of the velocity, even though different exponents had to be used in different velocity zones. The series of these exponents, as assumed, for example, in Ingalls' Ballistic Tables, gives a good idea of what takes place in the critical zone and elsewhere. They are given in the following table, where the first column indicates velocities, in feet per second, and the second column, the magnitude of the resistance:

For

$$\begin{aligned}
v &< 790, & R &= A_1 v^2;\\
790 &< v < 970, & R &= A_2 v^3;\\
970 &< v < 1230, & R &= A_3 v^6;\\
1230 &< v < 1370, & R &= A_4 v^3;\\
1370 &< v < 1800, & R &= A_5 v^2;\\
1800 &< v < 2600, & R &= A_6 v^{1.7};\\
2600 &< v < 3600, & R &= A_7 v^{1.55}.
\end{aligned}$$

The A's are constant coefficients, which are so chosen as to give a function that is continuous from zone to zone; its graph, however, has corners at the points where two zones come together.

The resistance function which is tabulated for use in applying the method described in the following pages, while agreeing approximately with the above, has a continuous derivative, in addition to being continuous itself; its graph is smooth, as well as uninterrupted. This is important in connection with the computation of differential variations. There is no need of a formula to represent it, and it can not be represented by any simple formula. It is to be understood that the limitations of the experimental data leave some room for discretion as to the best values to use.[1]

[1] The tabulated function G (v) is not directly proportional to the resistance, but has a factor v taken out, for reasons which appear in connection with the discussion of the differential equations.

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In what has been said so far, it is assumed that the air is at standard density and temperature. The question of changes in density will be taken up in section 4. Since a change in temperature alters the velocity of sound, it may be expected to have an influence on the critical portion of the resistance curve in the neighborhood of that velocity, apart from its effect on the density. There is very little experimental evidence on which to base a calculation of the temperature effect, but a reasonable assumption, making the resistance depend on the ratio of the velocity of the projectile to that of sound, has been found in particular cases to lead to the removal of discrepancies in the results of range firing which could not be traced to any other cause. No detailed discussion of the temperature correction is undertaken in the present pamphlet.

3. *Ballistic coefficient.*—It is implied in the preceding section not merely that the resistance varies in the general manner indicated in the case of any one projectile, but also that a single resistance function can be used in all cases. This does not mean that the resistance is the same for all projectiles, as it obviously is not; the assumption is, that its value can be found from the table by the introduction of a suitable constant of proportionality, characteristic of the projectile, and known as its "ballistic coefficient." More exactly, the quantity which is needed in computation is the retardation, rather than the resistance, the ratio of the two quantities involving the mass of the moving body. A projectile for which the retardation has the value indicated by the table is said to have ballistic coefficient unity. In other cases, the ballistic coefficient, commonly denoted by C, is the reciprocal of the ratio of the actual retardation to that given by the table. It is greater, therefore, as the retardation is less, and measures the power of the projectile to penetrate the air. For example, it is about 2 for a 75-mm. shell, and about 10 for a 14-inch shell.

Experimental firings with each particular projectile are necessary in order to determine the ballistic coefficient accurately enough for purposes of range table computation. It depends on the weight and caliber of the projectile, on the details as well as the general characteristics of its external form, and on the distribution of weight within its body. A projectile never keeps its axis strictly tangent to the path, and this has an important effect on the range; as present methods provide only for the calculation of head-on resistance, the effect of obliquity, or yaw, is taken up in the value of C, which is adjusted to agree with the observed range. Lack of complete knowledge of the dependence of the resistance on the shape of the projectile would make some such adjustment necessary, even if there were no question of yaw.

It is customary to write the ballistic coefficient in the form

$$C=\frac{w}{id^2},$$

in which w stands for the weight of the projectile in pounds, and d for its diameter in inches, while the quantity i, known as the coefficient of form, covers everything else, and has to be determined by experiment.[1] It is reasonable to suppose that the retardation will be directly proportional, and the ballistic coefficient inversely proportional, to the cross-section; and that the resistance will be independent of the weight, other things being equal, and the retardation therefore inversely proportional to the weight. This is the explanation of the factors w and d^2. The coefficient of form is then an indication of the extent to which expectations based on weight and caliber are realized in the actual design of the shell. Projectiles are still in use for which it is approximately 1, while for modern shell of the best design its value may be ½ or even less.

The assumption that the resistance at different velocities is proportionately the same for all projectiles would hardly be expected to hold with strict accuracy, and experiment confirms the existence of variations from it. The various types of shell in actual use, however, have enough general resemblance to each other so that it is found fairly satisfactory as a working hypothesis.

4. *Dependence of the resistance on the density of the air; the altitude function.*—It remains to consider the effect of variations from the standard uniform air density which has been assumed in the preceding sections, both the variation due to height and the variations at any particular point from time to time. It is natural to suppose that the resistance is directly proportional to the density, and there is no occasion for questioning the validity of this hypothesis in practice. The main question is that of the law of diminution of density with altitude. This is found to be given with fair accuracy by a single exponential formula up to a height of nine or ten kilometers, where a considerable deviation from the formula becomes apparent. The altitude function, like the velocity function, is used in tabular form, or at any rate may be so used, and it might be possible to construct a table which would be a little closer to the facts, on the average, even at moderate altitudes; but the gain would be slight in comparison with certain secondary advantages which come from the use of the formula. The exponential law is therefore used in ordinary computation. Comparatively few trajectories reach the height at which it begins to be considerably in error; and for ordinary trajectories it is good enough as it stands, when great accuracy is not required, while for the most accurate work it is necessary to apply differential corrections in any case for the variations of the day from whatever normal law is assumed, so that within moderate limits one is as good as another.

[1] In the method presented here, factors for wind, altitude, and air density are not included in C, as in the use of Ingall's Tables, for example, their effects being taken into account explicitly, as will be explained later; and there is no occasion for a so-called "integrating factor."

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CHAPTER I.

COMPUTATION OF TRAJECTORIES.

1. *Significance of the method of numerical integration.*—The method of computing ranges and other data pertaining to the flight of a projectile by means of the ballistic tables which have been used until recently as the basis for such computation gives results which are satisfactory for low angles of elevation. For high-angle fire, however, the results may be very inaccurate, and the mathematical theory back of the tables is so involved that it is difficult to trace the errors to their source and to make suitable compensation for them. A method has, therefore, come into use which is mathematically simple and straightforward in the highest degree. It is known as the method of numerical integration; its introduction in the form in which it is used in this country is due to Major F. R. Moulton. The treatment of the differential equations is based directly on the fundamental fact that a derivative is a rate of change, and that the value of a quantity can be computed step by step when its rate of change is known. The method is capable of giving any degree of accuracy which the physical data will permit, and can be adapted immediately to new problems as they arise. It is necessarily more laborious of application in particular cases, so long as no tables are in existence corresponding to those by which the other method is put into effect. Such tables are now in process of preparation, however, and the use of the principle of numerical integration seems destined to displace the older theory as the only method adequate to the further development of the science.

2. *The differential equations.*—The differential equations for the horizontal and vertical components of the motion are

$$\frac{dv_x}{dt} = -\mathrm{R}\cos\theta,$$

$$\frac{dv_y}{dt} = -\mathrm{R}\sin\theta - g.$$

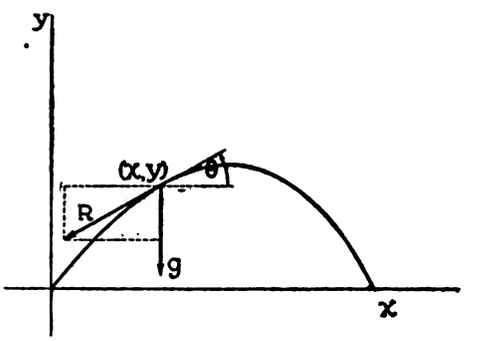

Fig. 1

In these equations x is the horizontal and y the vertical distance from the point of departure; v_x and v_y are the horizontal and vertical components of the velocity, equal to dx/dt and dy/dt, respectively; R is the retardation due to the resistance of the air, measured in feet (or meters) per second per second; θ is the angle which the tangent to the trajectory at any instant makes with the horizontal, reckoned as positive on the ascending and negative on the descending branch; and g is the acceleration of gravity. The components of R are proportional to $\cos\theta$ and $\sin\theta$, because the direction of the resistance is along the tangent. If v is the total velocity, with v_x and v_y for its components, $v_x = v\cos\theta$ and $v_y = v\sin\theta$, so that $\cos\theta$ and $\sin\theta$ in the equations may be replaced by v_x/v and v_y/v, respectively. Let R/v be denoted by E, so that

$$\mathrm{R} = \mathrm{E}\cdot v;$$

then

$$\frac{dv_x}{dt} = -\mathrm{E}\cdot v_x, \qquad \frac{dv_y}{dt} = -\mathrm{E}\cdot v_y - g,$$

or, if v_x and v_y are denoted by x' and y', and dv_x/dt and dv_y/dt by x'' and y'',

$$x'' = -\mathrm{E}x', \qquad y'' = -\mathrm{E}y' - g. \tag{1}$$

The equations are used in this form without further transformation.

3. *The tabulated resistance functions.*—The function E is assumed to have the form

$$\mathrm{E} = \frac{\mathrm{G}(v)\mathrm{H}(y)}{\mathrm{C}}$$

where G is a function of the velocity alone and H is a function of the altitude. The ballistic coefficient C is constant throughout the trajectory, and in general has a fixed value for any one projectile, except so far as minor modi-

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fications may be necessary in practice to secure the closest possible agreement with actual firings at different angles of elevation and different velocities.[1] Logarithmic tables of the G and H functions are attached. The tables are given in the form in which they have actually been used in recent work, and are in metric units because such units were employed in the preparation of range tables for use in France. The argument in the H table is simply y, measured in meters, while that of the G table is $v^2/100$, because the latter quantity is more readily computed than v itself when the components x' and y' are given. The value of H is the ratio of the density of the air at height y to that at the ground, according to the law of density assumed. In the table given here,[2] H is represented by the formula

$$H(y) = 10^{-.0000459y},$$

and its logarithm can be very easily calculated from this formula for values of y beyond the limits of the table.[3] There is no simple formula for G.

4. *Numerical example—Start of the computation.*—The procedure is best illustrated by a numerical example. A definite arrangement is followed here for the sake of clearness; but the details of the work can be varied in many ways, and every computer will be led by experience to introduce modifications which appeal to him as convenient. Suppose a projectile with ballistic coefficient 3.6 is fired (from a 155-mm. G. P. F. gun, for instance) with an initial velocity of 2,400 feet per second, or 731.5 meters per second, at an angle of departure of 30°. The initial values of x' and y' are found to be 633.5 and 365.8. The value of $v^2/100$ is 5351, to four figures. The corresponding value of log G, from the table, is 9.4515. Since $y=0$, H is simply equal to 1. The quantities Ex' and Ey' are found to have the values 49.78 and 28.74, and with the value 9.81 for g, it follows that $Ey'+g$ is 38.55. This means that when the projectile starts the horizontal component of its velocity is diminishing at the rate of 49.78 meters per second per second, and the y component of velocity is diminishing at the rate of 38.55 meters per second per second. If this rate of change were to continue throughout a whole second, the components of velocity at the end of the second would be reduced to 583.7 and 327.2 meters per second. (There is some advantage in computing the components of acceleration to hundreths of meters, even though the components of velocity are carried only to tenths.)

To take a smaller time interval, for reasons which will appear presently, the changes in the components of velocity during a quarter second would be 12.4 and 9.6, and the values at the end of this interval would be 621.1 and 356.2. While not entirely accurate, these give a good first approximation to the actual values. They can be used to compute approximately the retardation at the end of the quarter second. The corresponding value of $v^2/100$ is 5126, and that of log G is 9.4464. There is occasion now, however, for taking account of the height of the projectile above the ground. The approximate mean value of y' during the quarter second is 361.0, and the approximate height at the end of the quarter second is one-fourth of this, or 90.2. This leads to the value 9.9959 for log H. When the computation is carried through, the values 47.78 and 37.20 are found for Ex' and $Ey'+g$ at the end of the interval.

5. *Second approximation, first interval.*—The first approximate values of x' and y' for $t=\frac{1}{4}$ were computed on the assumption that the components of acceleration have the same values throughout the quarter second as at the beginning. A better approximation is obtained by using the mean between the values at the beginning and the approximate values at the end. On this assumption, the changes in x' and y' will be 12.2 and 9.5, respectively, during the quarter second, and the terminal values will be 621.3 and 356.3. The corresponding value of y is 90.3. The computation of Ex' and $Ey'+g$ is now repeated with these approximate values. It is found that $Ex'=47.80$, and $Ey'+g=37.21$. On recomputation of the average components of retardation for the interval, the same values as before are found for x', y', and y, and the train of successive approximations has been brought to a close. For $t=\frac{1}{4}$,

$$x'=621.3, \qquad y'=356.3, \qquad y=90.3.$$

The value of x is not needed at present.

6. *Second interval.*—It is possible now to go on to the next quarter second, from $t=\frac{1}{4}$ to $t=\frac{1}{2}$. This computation can be started to better advantage than that for the first interval, as not only the values of the retardation components at the beginning of the interval are known now, but also the amounts by which they have changed during the preceding quarter second, so that the values of the components can be estimated with fair accuracy for $t=\frac{1}{2}$. If the change in the second interval is assumed to be the same as in the first, the approximations $Ex'=45.82$ and $Ey'+g=35.87$ are obtained for $t=\frac{1}{2}$; 609.6 and 347.2 are the approximate values of x' and y' for the same instant, obtained by using the average retardations for the quarter second, and 178.3 is the corresponding value of y. From these are computed the improved values—

$$Ex'=45.94, \qquad Ey'+g=35.98.$$

Since these lead back to the x', y', and y already used, further computation for the interval is unnecessary.

[1] Since the dependence of the resistance on the altitude is taken into account directly in the course of the computation, there is no altitude factor in the ballistic coefficient, as there is in the method used with the older ballistic tables. Wind and variations in air density are also taken account of separately; see Chap. II.

[2] There is no occasion for using more than four places in computation; five places are tabulated because the logarithms to five places are exact, on the basis of the formula used.

[3] An alternative method is suggested at the head of the table.

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7. *Summary of the general procedure.*—By continuing the process, it is possible to calculate the values of y, x', y', x'', and y'', step by step, along the whole trajectory.[1] These five quantities are so related through the differential equations that it is necessary to carry them all along together in the calculation. The coordinate x, on the other hand, does not enter into the resistance functions, and it can be found by summation of the values of x', after the rest of the work has been done.

It has been assumed that the change in x, y, x', or y' during any one interval can be found by using the mean of the values of x', y', x'', or y'', respectively, at the beginning and at the end of the interval, as an average rate of change for the whole interval. Quarter seconds are used, in the present illustration, in order that this assumption may be sufficiently accurate. No fixed rule can be given for the choice of the best interval of time. It varies with the velocity and with the ballistic coefficient, and a decision will be reached in any particular case by trial and by experience. Half seconds would have served nearly or quite as well in the present instance, while the use of whole seconds would have been questionable. It will be shown presently that it is possible to use longer intervals, by taking account of second differences, and that different intervals can be used to advantage for different parts of a single trajectory.

For any one interval the method is one of successive approximations, in principle, though the first computation will usually be so nearly accurate that it will not be necessary to carry through a second one, except at the very beginning of a trajectory. Let the interval in question be that from $t=t_1$ to $t=t_1+\Delta t$. At the start, all five of the quantities y, x', y', x'', y'', are known for $t=t_1$, and none of them are known for $t=t_1+\Delta t$. If the components of the acceleration were known for the latter instant, it would be possible to find y, x', and y', from their initial values and rates of change. If y, x', and y' were known, the values of x'' and y'' could be found by substitution in the differential equations. The procedure actually followed here is, to estimate the terminal values of x'' and y'', by extrapolation from those already calculated for earlier instants, to use these approximate values for the calculation of y, x', and y', and then to compute x'' and y'' once more from the latter quantities, by means of the differential equations and the tabulated resistance functions.[2] The last computation checks the accuracy of the original approximation to x'' and y''. The agreement will seldom be perfect to the second place of decimals, and the first approximate values are replaced by the new ones. Recomputation of x', y', and y, however, will show little if any change from the values previously used. If there should be any appreciable discrepancy the work would be repeated until agreement is reached. In the case of a change of one-tenth of a unit in x' and y', while the new values should be put in place of the old ones as final determinations of the quantities in question, it is not necessary to repeat the computation of the accelerations by the G and H tables. It will be found even that somewhat larger changes in x', y', and y are generally without appreciable effect on x'' and y''. The ultimate convergence of the process of successive approximations can be demonstrated by modern methods in mathematical analysis. Such a proof would not be in place here, however. A little experience will leave no doubt as to the fact of the convergence.

It goes without saying that reliance can not be placed on the last figure retained. In a long computation the accumulation of neglected decimals may lead to an error of many units in the last place, affecting even the preceding figure. A difference of one-tenth of a meter in the horizontal component of velocity at the beginning of a 50-second trajectory, for example, would correspond directly to a difference of five meters in the range. It is for this reason that more figures are used in the computation than would have any significance in the final result. Hundredths of meters are used in the accelerations for the purpose of diminishing the uncertainty of the tenths of meters in the other quantities.[3]

Exercise.—Start the computation of a trajectory with

$$C=3.6, \qquad v_0=2{,}400 \text{ foot-seconds}, \qquad \phi=20°.$$

(The details of the computation should be kept on file with a view to completing the trajectory later.)

8. *Use of second differences, numerical example.*—The computation of a whole trajectory by the method outlined above, with quarter-second or even with half-second intervals throughout, would be a long process. As has been indicated already, it is possible to use longer intervals and to lighten the work correspondingly by taking account of second differences. The manner of doing this will be illustrated numerically, and then the derivation of the formula will be explained.

---

[1] It will be seen subsequently that the work can be much shortened in practice.

[2] It is equally possible to begin by estimating the velocity components and the altitude, to compute the accelerations from them, and then to come back to x', y', and y. A device known as the "formula for integrating ahead," introduced by Mr. J. J. Arnaud of the Ordnance Department, has been used extensively in practice, but will not be explained in detail here.

[3] It will happen many times in the course of the work that a number in which the last figure is odd has to be divided by 2, or a number is obtained whose last figure is otherwise indeterminate. In such cases where an arbitrary choice has to be made, a convenient rule is to take the even number, unless there is some reason to the contrary.

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On further computation of the trajectory which was started in an earlier paragraph, with the parameters $v_0 = 731.5$ meter-seconds, $\phi = 30°$, $C = 3.60$, the following table of values is found:[1]

$t=0$	$y=$ 0.0	$y'=365.8$		
			36.0	
1	347.5	329.8		4.3
			31.7	
2	661.2	298.1		3.5
			28.2	
3		269.9		

The first column gives the time, measured in seconds from the start; the second column gives the corresponding values of y, except that for $t=3$, which is to be determined; and the third gives the values of y'. The fourth column contains the numerical values of the first differences[2] of the quantities y', obtained by subtracting each entry in the y' column from the one above it. The fifth column gives the second differences of y', that is, the first differences of the first differences.

Suppose for the moment that second differences were to be neglected. Then the average rate of change of y during the interval from $t=2$ to $t=3$ would be the average of 298.1 and 269.9, that is, 284.0. Since the length of the interval is one second, the total change would be numerically the same as the average rate of change, and the value of y at the end of the interval would be $661.2+284.0=945.2$. To make the correction for second differences, it is necessary merely to diminish the mean value 284.0 by one-twelfth of the second difference, 3.5, which stands in the same horizontal line with $t=2$. That is, for the average rate of change, or total change during one second, 284.0 is to be replaced by $284.0-0.3=283.7$. The corrected value of y for $t=3$ is $661.2+283.7=944.9$. It is seen that the labor of applying the correction is negligible.

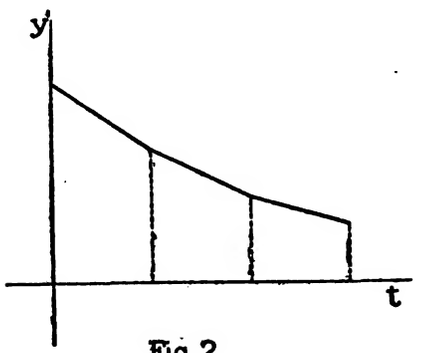

Fig. 2

The reason why the first average needs to be diminished becomes apparent if the value of y is written as an integral, $y=\int y'\,dt$, and the integral is interpreted as the area under a curve. Let y' be plotted against t for $t=0$, 1, 2, 3. In a figure drawn accurately to scale, the four plotted points would be nearly in a straight line. It is clear, however, that the lines joining successive points are less and less inclined to the horizontal, because the first differences are dimishing in size. This is exaggerated in the accompanying figure, in order to bring out what is essential. Using the mean of the initial and terminal values as average over an interval amounts to calculating the area of the trapezoid under a segment of the broken line, and this area is evidently greater than that under a smooth curve through the plotted points, such as would represent the actual continuous changes of y'. The explanation of the coefficient 1/12 will be brought out in the next section.

The table above, continued a little further, would read as follows:

$t=2$	$y=$ 661.2		$y'=$ 298.1		
		283.7	(284.0)	28.2	
3	944.9		269.9		2.9
		257.0	(257.2)	25.3	
4	1201.9		244.6		2.5
		233.0	(233.2)	22.8	
5	1434.9		221.8		2.0
		211.2	(211.4)	20.8	
6	1646.1		201.0		

The third column here shows the first differences of y. Each of the numbers in parentheses in the fourth column is the mean of the numbers standing above and below it, and comparison of these numbers with those in the third column shows how the second differences have had an effect each time. In a very large proportion of cases in practice, to be sure, the effect of the second differences will not appear in the figures retained, and the calculation of them will have served merely to give assurance that they are negligible. It is understood, of course, that the figures above represent only a part of the whole computation, and that all five of the quantities y, x', y', x'', y'' are computed for any one second before passing on to the next.[3]

[1] A more extensive table, with an indication of the time intervals used in different parts of the work, is given in sec. 11.

[2] The question of the algebraic signs of these quantities will be taken up later.

[3] In connection with the results of computation set down here, and others to be given further on, it is to be recalled, as mentioned once already, that there are numerous cases in which a more or less arbitrary choice has to be made between two possible values for the last figure of a number, and slight differences in the results of calculations by two different computers do not necessarily imply a mistake on the part of either. Such slight differences are even more to be expected if the two computers have not used the same time intervals throughout the work.

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If the interval of time is something other than one second, the average rate of change during the interval is still to be found by taking the mean of the initial and terminal values, and subtracting one-twelfth of the second difference. The average rate of change so obtained is to be multiplied by the length of the interval to obtain the total change.

In the case of two-second intervals it is easier, in place of the mean, to calculate twice the mean directly, by adding the initial and terminal values. The effect of the second difference for the two seconds is then found by taking one-sixth of the tabulated second difference. This is illustrated by the following figures, still referring to the same trajectory as before:

$t=12$	$y=2546.4$		$y'=106.1$		2.9
		187.4	(187.9)	24.3	
14	2733.8		81.8		1.9
		140.9	(141.2)	22.4	
16	2874.7		59.4		

The second difference 2.9 in the first line is found from earlier entries not set down here. Each number in parentheses now is the sum of the numbers above and below it, and is diminished by one-sixth of the corresponding second difference of y' to give the difference of y at the left.[1]

In correcting for second differences in general, the algebraic sign of the correction will, of course, depend upon circumstances, and will not always be the same as in the example treated here. In cases of doubt, the question may be decided by drawing a figure as above, or by referring to the detailed reasoning of the next section.

Exercise.—From the $x'-$column in the table of section 11, and the values of x for $t=3$ and for $t=14$, calculate x for $t=4, 5, 6, \ldots$ and for $t=16, 18, 20, \ldots$ and compare the values so found with those given in the $x-$column.

9. *Derivation of the formula for the use of second differences.*—Let z be a tabulated function of t, and let z_1, z_2, z_3 be the values of z corresponding to $t=t_1, t_2, t_3$, respectively.

Let

$$a_1=z_2-z_1, \quad a_2=z_3-z_2, \quad b_1=a_2-a_1.$$

These quantities may be arranged in tabular form as follows:

$$\begin{array}{cccc} t_1 & z_1 & & \\ & & a_1 & \\ t_2 & z_2 & & b_1 \\ & & a_2 & \\ t_3 & z_3 & & \end{array}$$

It is desired to compute from them the value of

$$\int_{t_2}^{t_3} z\,dt.$$

In the first numerical illustration worked out in the preceding section, for example,

$$t_1=1, \quad t_2=2, \quad t_3=3, \quad z_1=329.8, \quad z_2=298.1, \quad z_3=269.9, \quad a_1=-31.7, \quad a_2=-28.2, \quad b_1=3.5.$$

It is of course important that all the quantities be taken with the proper algebraic signs; if this is done, the formulas will hold in all cases.

It is evident geometrically, as was indicated in connection with the figure in the preceding section, that the use of $\frac{1}{2}(z_2+z_3)$ as an average value of z from t_2 to t_3, in calculating the integral, amounts to assuming that the graph of z in the interval can be represented by a straight line. It will be worth while to work out the same fact algebraically, as this will help to bring out the meaning of the corresponding formula with second differences.

Let[2]

$$f_1(t)=z_3+\frac{z_2-z_3}{t_3-t_2}(t_3-t).$$

This is a function of the first degree in the variable t, so that its graph is a straight line. On substitution of the values $t=t_3$ and $t=t_2$ successively, it is found that

$$f_1(t_3)=z_3, \quad f_1(t_2)=z_2.$$

That is, $f_1(t)$ is a formula of linear interpolation for z in the interval from t_2 to t_3. The integral $\int f_1(t)\,dt$ gives a corresponding approximation to $\int z\,dt$ over the interval.

[1] It is to be pointed out that the special method indicated for two-second intervals may improve the accuracy of the work. The mean of 106.1 and 81.8, to one place of decimals, is 94.0; a twelfth of 2.9, to the same degree of accuracy, is 0.2; the average rate of change per second is found to be $94.0-0.2=93.8$, and doubling this, for two seconds, gives 187.6 (instead of 187.4 above) as the first difference of y. The errors committed in dividing and dropping extra decimals are magnified on multiplying again. Similar remarks are applicable, with even greater force, in case four-second intervals are employed.

[2] It will be seen that while t, z, a, b, etc., have corresponding subscripts, the subscripts attached to the letter f, here and later, have a different significance.

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To evaluate the former integral,

$$\int_{t_2}^{t_3}(t_3-t)dt=-\frac{1}{2}\Big[(t_3-t)^2\Big]_{t=t_2}^{t=t_3}=\frac{1}{2}(t_3-t_2)^2,$$

$$\int_{t_2}^{t_3}f_1(t)\,dt=z_3(t_3-t_2)+\frac{1}{2}(z_2-z_3)(t_3-t_2)$$

$$=\frac{1}{2}(z_2+z_3)\,(t_3-t_2).$$

This is exactly the result obtained by regarding $\frac{1}{2}(z_2+z_3)$ as the average value of z from $t=t_2$ to $t=t_3$. Since $z_2=z_3-a_2$, the result can also be written in the form

$$\int_{t_2}^{t_3}f_1(t)dt=\Big(z_3-\frac{1}{2}a_2\Big)(t_3-t_2),$$

so that it is a formula of integration involving first differences.

The introduction of second differences, in the manner indicated in the preceding section, amounts to replacing the function $f_1(t)$, of the first degree, correct for two particular values of t, by a function of the second degree, correct for three values of t. Geometrically, the straight line through two points is replaced by a parabola passing through three of the given points.

Let

$$f_2(t)=z_3+\frac{z_2-z_3}{t_3-t_2}(t_3-t)-\frac{1}{t_2-t_1}\left[\frac{z_2-z_3}{t_3-t_2}-\frac{z_1-z_3}{t_3-t_1}\right](t_3-t)\,(t_2-t).$$

This is a function of the second degree in t. By substituting the values t_3, t_2, and t_1 for t, it is found that

$$f_2(t_3)=z_3,\qquad f_2(t_2)=z_2,\qquad f_2(t_1)=z_1.$$

In practice, t_1, t_2, and t_3 will be equally spaced, and the formulas can be simplified by taking this into account. Let $t_3-t_2=t_2-t_1=h$. With this notation, and with the definitions of a_1, a_2, and b_1 given at the beginning of the section,

(2) $$f_2(t)=z_3-\frac{a_2}{h}(t_3-t)+\frac{1}{h}\left[\frac{a_2}{h}-\frac{a_1+a_2}{2h}\right](t_3-t)\,(t_2-t)=z_3-\frac{a_2}{h}(t_3-t)+\frac{b_1}{2h^2}(t_3-t)\,(t_2-t).$$

Since

$$\int_{t_2}^{t_3}(t_3-t)dt=\frac{1}{2}h^2$$

and[1]

$$\int_{t_2}^{t_3}(t_3-t)\,(t_2-t)dt=-\frac{1}{6}h^3,$$

it follows that

$$\int_{t_2}^{t_3}f_2(t)\,dt=h\Big(z_3-\frac{1}{2}a_2-\frac{1}{12}b_1\Big).$$

As $z_3-\frac{1}{2}a_2=\frac{1}{2}(z_2+z_3)$, this gives the rule which was followed in the preceding section.

In the same way, formulas can be worked out containing differences up to any order that may be desired. The next, obtained from an interpolating function of the third degree, would be

$$h\Big(z_3-\frac{1}{2}a_2-\frac{1}{12}b_2-\frac{1}{24}c_1\Big).$$

It is understood that z_1, z_2, z_3, z_4 are the values of z corresponding to t_1, t_2, t_3, t_4, that

$$t_2-t_1=t_3-t_2=t_4-t_3=h,$$

and that

$$z_2-z_1=a_1,\qquad z_3-z_2=a_2,\qquad z_4-z_3=a_3,$$
$$a_2-a_1=b_1,\qquad a_3-a_2=b_2,\qquad b_2-b_1=c_1.$$

[1] One way of performing this integration is as follows: Let $s=t-t_2$. Then $ds=dt$, and

$$\int_{t_2}^{t_3}(t_3-t)\,(t_2-t)dt=-\int_0^h(h-s)sds=-\Big[\tfrac{1}{2}hs^2-\tfrac{1}{3}s^3\Big]_{s=0}^{s=h}=-\tfrac{1}{6}h^3.$$

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In computing trajectories, however, the use of differences of higher order than the second is not recommended in general. If the third differences are too large to be neglected, the remedy is to be found in using a smaller interval of time. The formula just given is useful as indicating in any particular case whether the third differences are negligible or not.

Exercise.—If

$$f_3(t)=z_4-\frac{a_3}{h}(t_4-t)+\frac{b_2}{2h^2}(t_4-t)\ (t_3-t)-\frac{c_1}{6h^3}(t_4-t)\ (t_3-t)\ (t_2-t),$$

show that

$$f_3(t_4)=z_4, \qquad f_3(t_3)=z_3, \qquad f_3(t_2)=z_2, \qquad f_3(t_1)=z_1.$$

Exercise.—Show that

$$\int_{t_3}^{t_4} f_3(t)dt=h(z_4-\tfrac{1}{2}a_3-\tfrac{1}{12}b_2-\tfrac{1}{24}c_1).$$

Exercise.—If

$$\varphi_2(t)=z_1+\frac{a_1}{h}(t-t_1)+\frac{b_1}{2h^2}(t-t_1)\ (t-t_2),$$

show that

$$\varphi_2(t_1)=z_1, \qquad \varphi_2(t_2)=z_2, \qquad \varphi_2(t_3)=z_3.$$

Exercise.—If

$$\varphi_3(t)=z_1+\frac{a_1}{h}(t-t_1)+\frac{b_1}{2h^2}(t-t_1)\ (t-t_2)+\frac{c_1}{6h^3}(t-t_1)\ (t-t_2)\ (t-t_3),$$

show that

$$\varphi_3(t_1)=z_1, \qquad \varphi_3(t_2)=z_2, \qquad \varphi_3(t_3)=z_3, \qquad \varphi_3(t_4)=z_4.$$

Exercise.—Find a function of the fourth degree, agreeing in value with z at five points. (Notice that, in the denominators of the preceding formulas, $2=2\cdot 1$, and $6=3\cdot 2\cdot 1$.)

10. *Change of length of interval.*—As the computation of a trajectory proceeds, it will be found that the differences run more smoothly, so that it becomes practicable in many cases to use a longer time interval. It is most convenient to take for the increased interval the double of that which has been used. The change from one length to the other does not call for the use of any special methods. If the various quantities have been computed for $t=8$, 9, 10, 11, 12, for example, the values for $t=8$, 10, 12, form a basis for continuing the computation with two-second intervals. It is merely necessary to exercise a little closer attention than is required when the work is proceeding by an uninterrupted routine.

There is particular occasion for changing the length of interval in the early stages of a computation, if very short intervals are employed at the outset, for the sake of rendering second differences negligible. When second differences are taken into account, the completion of the computation for any one instant involves the use of a second difference derived from the results of computation for earlier instants. At the start, no such second difference is available. The difficulty may be met either by starting with intervals so short that second differences are unnecessary, as in the numerical example above, or by estimating a second difference on the basis of a preliminary rough computation. An experienced computer will probably prefer the latter method; the former has been chosen for detailed explanation, as a little more direct. It may be emphasized once more that experience will suggest numerous modifications of procedure which will commend themselves more or less forcibly in practice, though an attempt to treat them in a general discussion would only obscure the underlying principles.

It may happen that as the velocity increases on the descending branch of a trajectory, the differences increase to such an extent that a return to shorter intervals becomes desirable. This will call for a certain amount of interpolation among the last quantities computed before making the change. The function $f_2(t)$ of the preceding section in the form (2) may be used here as an interpolating function taking account of second differences. For the middle point of the interval from t_2 to t_3, it gives

$$f_2\left(\frac{t_2+t_3}{2}\right)=z_3-\tfrac{1}{2}a_2-\tfrac{1}{8}b_1.$$

Care must be taken to compute the various terms with the proper algebraic signs. The first two give the average of z_2 and z_3, and the third is a correction for second differences.

Exercise.—In the table of section 11 compute the entries for $t=3$ and for $t=14$, regarding the preceding entries as known.

11. *Numerical data for a specimen trajectory.*—By way of illustration, the values of x, y, x', y', Ex', and $Ey'+g$ are given for the whole of the trajectory which has been used in the preceding numerical examples. It is understood

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that the figures given here represent only a part of the entire computation, and that some sort of regular computing form should be used, both for the integration by means of difference formulas, as explained above, and for the calculation of the components of acceleration by means of the G and H tables. It should also be mentioned once more that accidental differences in the choice of the last figure would be likely to accumulate to some extent on repetition of the work.

The computation in this example was carried out by quarter-second intervals from $t=0$ to $t=1$, by half seconds from $t=1$ to $t=2$, by whole seconds from $t=2$ to $t=12$, and by two-second intervals from $t=12$ to the end. In the choice of intervals the tendency has been to err on the side of taking them shorter than was strictly necessary.

TABLE OF COORDINATES AND COMPONENTS OF VELOCITY AND ACCELERATION FOR—

$v_0=2{,}400$ foot-seconds$=731.5$ meter-seconds, $\phi=30°$, $C=3.60$.

t	x	x'	Ex'	y	y'	$Ey'+g$
0	0	633.5	49.78	0	365.8	38.55
¼	156.8	621.3	47.80	90.3	356.3	37.21
½	310.6	609.6	45.94	178.3	347.2	35.98
¾	461.6	598.3	44.19	264.0	338.4	34.81
1	609.8	587.5	42.56	347.5	329.8	33.70
1½	898.3	567.0	39.53	508.3	313.5	31.66
2	1176.9	547.9	36.78	661.2	298.1	29.82
3	1707.3	513.6	32.04	944.9	269.9	26.65
4	2205.5	483.6	28.02	1201.9	244.6	23.99
5	2675.6	457.3	24.62	1434.9	221.8	21.75
6	3121.1	434.2	21.67	1646.1	201.0	19.84
7	3544.9	413.9	19.08	1837.5	182.0	18.20
8	3949.7	396.0	16.81	2010.6	164.5	16.79
9	4337.6	380.2	14.76	2166.9	148.3	15.57
10	4710.7	366.4	12.92	2307.6	133.3	14.51
11	5071.0	354.3	11.24	2433.8	119.3	13.59
12	5419.8	343.8	9.70	2546.4	106.1	12.80
14	6089.6	327.0	7.18	2733.8	81.8	11.61
16	6730.4	314.5	5.44	2874.7	59.4	10.84
18	7349.2	304.8	4.38	2972.1	38.2	10.36
20	7950.5	296.8	3.72	3028.0	17.8	10.03
22	8536.9	289.8	3.31	3043.7	− 2.0	9.79
24	9110.1	283.5	3.04	3020.2	− 21.4	9.58
26	9671.1	277.6	2.86	2958.3	− 40.4	9.39
28	10220.7	272.0	2.78	2858.9	− 59.0	9.21
30	10759.2	266.5	2.73	2722.6	− 77.2	9.02
32	11286.7	261.0	2.73	2550.4	− 95.0	8.82
34	11803.2	255.5	2.77	2342.9	−112.4	8.59
36	12308.6	249.9	2.85	2101.1	−129.3	8.34
38	12802.6	244.1	2.96	1826.0	−145.7	8.04
40	13284.7	238.0	3.10	1518.7	−161.5	7.71
42	13754.3	231.6	3.26	1180.6	−176.5	7.32
44	14210.9	224.9	3.44	813.3	−190.7	6.89
46	14653.7	217.8	3.63	418.4	−204.0	6.41
48	15081.9	210.4	3.82	−2.1	−216.3	5.88

12. *Time of flight, range, and terminal conditions.*—It will be seen from the table above that the time of flight for the trajectory under discussion is almost exactly 48 seconds. More closely, since the projectile would be 2.1 meters below the level of the ground, if it continued its flight without interruption for 48 seconds, and since the vertical component of its velocity at the end is 216.3 meters per second, the time of reaching the level of the ground is $48-2.1/216.3=48-.01=47.99$ seconds.

The range and the components of the terminal velocity are now found as the values of x, x', and y', respectively, corresponding to $t=47.99$. The range is $15081.9-.01\times 210.4$, or, to whole meters,

$$X=15080.$$

For the velocity components,

$$x'_\omega=210.4+.01\times 3.82=210.4,$$
$$-y'_\omega=216.3-.01\times 5.88=216.2.$$

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From these can be found the terminal velocity and the angle of fall:

$$v_\omega = \sqrt{210.4^2 + 216.2^2} = 301.7,$$

$$\tan \omega = -y'_\omega / x'_\omega, \qquad \omega = 45^\circ.78 = 45^\circ\ 47'.$$

The calculation is simplified in the example just worked out by the accident that the instant of fall very nearly coincides with one of the even seconds used in the trajectory computation, so that the rates of change of the various quantities at the instant $t = 48$ can be used directly as average rates of change over the interval between the time when $y = 0$ and the end of the forty-eighth second. If the projectile had happened to reach the ground nearer the middle of one of the original intervals, greater accuracy would have been obtained by a slightly more elaborate method. To illustrate the same principle, let it be required to find for the present trajectory the time and the horizontal distance to the point where the projectile in its fall reaches a height of 150 meters above the ground. This is a height of 152.1 meters above the point reached at the end of the forty-eighth second, so that, as a first approximation, $t = 48 - 152.1/216.3 = 47.30$. The average vertical velocity from $t = 47.30$ to $t = 48$, obtained by linear interpolation in the y' — column, is 214.1, and the corrected time is $t = 48 - 152.1/214.1 = 47.29$. For the horizontal distance, the average horizontal velocity from $t = 47.29$ to $t = 48$ is 211.7, and

$$x = 15081.9 - .71 \times 211.7 = 14932,$$

to whole meters.

It would be possible to bring higher differences into the work, but if the correction is always reckoned from the nearest value of t used in the trajectory computation, forward or backward, as the case may be, the interval involved in the correction will be at most half of one of the intervals used in the original computation, and the method just indicated will be accurate enough. Moreover, there is no need of so great accuracy here as is required in the case of a step that is to be repeated many times, with consequent accumulation of whatever errors may be committed. It may be mentioned that the whole problem of interpolation at the end of the trajectory could be handled by means of the interpolation formulas discussed in section 9.

Exercise.—Find the value of v when $y = 150$, in the trajectory above.

Exercise.—Find the values of t, x, and v when $y = 200$.

13. *Maximum ordinate.*—The highest point of the trajectory is reached when $y' = 0$, for a value of t between 20 and 22. By linear interpolation between the values $y' = 17.8$ for $t = 20$ and $y' = -2.0$ for $t = 22$, it is found that $y' = 0$ for $t = 21.80$. There is no need of knowing the maximum ordinate with extreme accuracy, and this approximation is good enough. The distance that the projectile rises or falls during a short interval from the time of passing the top of the trajectory may be found either by the formula $\frac{1}{2}\, g\, (\Delta t)^2$, where Δt is the length of the interval, or by using an average velocity taken from the y' column; the vertical acceleration is so nearly g, and the average vertical velocity is so nearly $\frac{1}{2} g \Delta t$, for such an interval, that both methods give essentially the same result. In the present case the distance through which the projectile falls during the 0.20 second between the time of passing the top and the instant $t = 22$ is $\frac{1}{2} g\ (0.20)^2 = 0.2$ meter, so that, to the nearest meter, the maximum ordinate is the same as the ordinate for $t = 22$, namely, 3,044 meters. If the highest point were reached near the middle of an interval, the difference between the maximum ordinate and the height at either end of the interval would be appreciable.

14. *Modified form of the differential equations.*—Of the numerous transformations that have been proposed for the differential equations (1) of section 2, one of the most advantageous for use in connection with the method of numerical integration is that suggested by Capt. A. A. Bennett. It depends on the introduction of a new variable, defined by the equation

$$\sigma = \frac{y'}{g x'},$$

and is known as the tangent-reciprocal method. Accents are used throughout to indicate time derivatives, as before.

Direct differentiation of σ, with substitution of the values of x'' and y'' from (1), gives

$$\sigma' = \frac{1}{g} \cdot \frac{x'y'' - y'x''}{x'^2} = \frac{1}{g} \cdot \frac{-\mathrm{E}x'y' - gx' + \mathrm{E}x'y'}{x'^2} = -\frac{1}{x'}.$$

By a second differentiation,

$$\sigma'' = \frac{x''}{x'^2} = -\frac{\mathrm{E}x'}{x'^2} = \mathrm{E}\sigma'.$$

The equation

$$\sigma'' = \mathrm{E}\sigma'$$

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can be dealt with by the method of numerical integration in essentially the same way as the equations (1). It is simpler than the original pair of equations, but has the disadvantage that it does not give directly the quantities that are desired. When σ' and σ have been determined, the components of the velocity are obtained by means of the relations

$$x' = -\frac{1}{\sigma'} \qquad y' = -\frac{g\sigma}{\sigma'},$$

and x and y can be found from the latter quantities by integration.

The remark made more than once with regard to the integration of the equations (1), to the effect that there is much room for discussion as to the best arrangement of the details, applies even more forcibly to the modified equations, which, up to the present time, have not been studied so carefully nor used so extensively. For the sake of comparison, the trajectory already discussed at length was worked out again on the basis of the new equations, with steps as nearly as possible parallel to those previously taken, though this would certainly not be the best arrangement in using the new equations on their own merits. The results of the computation are given in the table below. It will be seen that $E\sigma'$ changes much more slowly and regularly than Ex' or $Ey'+g$; this smoothness, particularly at the beginning of the trajectory, is one of the chief advantages of the transformed equations. The discrepancies betwen the table of this section and that of section 11 are of the order of magnitude that should be expected, in view of the length of the computation and the opportunity for the accumulation of neglected decimals.

TABLE OF COORDINATES AND OTHER VARIABLES FOR—

v_0=2,400 foot-seconds=731.5 meter-seconds, ϕ=30°, C=3.60.

[COMPUTED BY MEANS OF THE TANGENT-RECIPROCAL EQUATIONS.]

t	x	x'	y	y'	σ	$-\sigma'$	$-E\sigma'$
0	0.0	633.5	0.0	365.8	.05884	.0015783	.0001240
¼	156.8	621.3	90.2	356.2	5844	16093	1238
½	310.7	609.7	178.1	347.1	5803	16402	1236
¾	461.7	598.4	263.8	338.3	5762	16711	1234
1	609.9	587.5	347.3	329.7	5720	17019	1233
1½	898.4	567.0	508.1	313.4	5633	17635	1229
2	1177.1	547.9	660.9	297.9	5543	18249	1226
3	1707.4	513.5	944.4	269.8	5354	19469	1214
4	2205.5	483.5	1201.3	244.4	5153	.0020675	.0001199
5	2675.6	457.3	1434.1	221.6	4940	.002187	.000118
6	3121.1	434.3	1645.1	200.9	4715	2303	115
7	3545.0	413.9	1836.4	181.9	4479	2416	111
8	3949.8	396.1	2009.5	164.4	4232	2525	107
9	4337.8	380.4	2165.8	148.4	3976	2629	102
10	4711.0	366.5	2306.6	133.4	3708	2728	96
11	5071.3	354.5	2432.9	119.3	3431	2821	90
12	5420.4	343.9	2545.5	106.1	3145	2907	82
14	6090.6	327.3	2732.9	81.8	2548	3056	67
16	6732.0	314.8	2873.8	59.4	1924	3177	55
18	7351.4	305.1	2971.2	38.2	1278	3278	47
20	7953.2	297.0	3027.2	17.9	613	3367	42
22	8540.1	290.1	3043.0	− 2.0	− 69	3448	39
24	9113.9	283.8	3019.6	− 21.3	− 766	3525	38
26	9675.5	277.8	2958.0	− 40.3	− 1479	3600	37
28	10225.3	272.0	2858.7	− 58.9	− 2206	3675	38
30	10763.8	266.6	2722.6	− 77.1	− 2949	3752	39
32	11291.4	261.0	2550.5	− 95.0	− 3707	3831	40
34	11808.0	255.6	2343.0	−112.4	− 4481	3913	42
36	12313.6	250.0	2101.2	−129.3	− 5272	4001	46
38	12807.7	244.0	1826.2	−145.6	− 6082	4097	50
40	13289.7	238.0	1519.1	−161.4	− 6912	4202	55
42	13759.4	231.6	1181.2	−176.4	− 7764	4318	61
44	14215.8	224.8	814.1	−190.6	− 8640	4447	68
46	14658.4	217.8	419.3	−204.0	− 9544	4591	76
48	15086.7	210.4	− 1.1	−216.2	−10478	4753	87

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15. *Checks; Simpson's Rule.*—A convenient and useful check on the integrations performed in the course of the work, whether with the equations of section 2 or with those of section 14, is given by the formula of summation known as Simpson's Rule. In the notation of section 9, if it is desired to find the value of $\int z dt$ over the interval from t_1 to t_n, where n is any *odd* number (so that the number of parts into which the interval is subdivided is even), an approximate value of the integral is

(3) $$\tfrac{1}{3}h\,(z_1+4z_2+2z_3+4z_4+2z_5+\ \ldots\ +2z_{n-2}+4z_{n-1}+z_n).$$

For the sum in parentheses, the first and last values of z are each counted once, the third, fifth, etc., are each counted twice, and the second, fourth, etc., four times. The sum is multiplied by the length of the interval between successive values of t, and divided by 3. It will be seen presently that this is essentially a formula taking account of second differences. Applied to the value of y' in the table of section 11, from $t=2$ to $t=12$, for example, it gives 1,885.4 for the increase in y during the 10 seconds, as compared with $2,546.4-661.2=1,885.2$, from the entries in the y-column. Exact agreement is not to be expected, partly because of neglected decimals, and partly because, while second differences are taken into account both in the original computation and in the check, they are not taken into account in exactly the same way.

To prove the formula, let the function $f_2(t)$ of section 9, in the form (2), be integrated from t_1 to t_3 (not merely from t_2 to t_3, as before). It is found that—

$$\int_{t_1}^{t_3}(t_3-t)dt=2h^2,$$

$$\int_{t_1}^{t_3}(t_3-t)(t_2-t)dt=\tfrac{2}{3}h^3,$$

$$\int_{t_1}^{t_3}f_2(t)dt=h(2z_2-2a_2-\tfrac{1}{3}b_1).$$

Since $a_2=z_3-z_2$ and $b_1=a_2-a_1=z_1-2z_2+z_3$, the last equation is equivalent to

$$\int_{t_1}^{t_3}f_2(t)dt=\tfrac{1}{3}h(z_1+4z_2+z_3).$$

This is a general formula for the area under the parabolic interpolating curve determined by three given points with equally spaced abscissas. The area under the parabola determined by the points corresponding to t_3, t_4, and t_5 is

$$\tfrac{1}{3}h(z_3+4z_4+z_5),$$

and so on. If the areas corresponding to the successive double intervals are added, the expression (3) is obtained.

A corresponding formula using only first differences is found by passing straight lines through pairs of successive points, and adding the areas of the trapezoids under these lines. This corresponds to the use of the function $f_1(t)$ in place of $f_2(t)$. The result is called the *trapezoid formula*, from the manner of its formation. The area of the first trapezoid is $\frac{1}{2}h(z_1+z_2)$, that of the second, $\frac{1}{2}h(z_2+z_3)$, and so on. The approximate expression for the integral from t_1 to t_n is

$$\tfrac{1}{2}h\,(z_1+2z_2+2z_3+2z_4+\ \ldots\ +2z_{n-1}+z_n).$$

In this case n may be either odd or even. The trapezoid formula is naturally less accurate than Simpson's Rule, for checking a computation in which second differences were used, but is just as good, and more convenient, if only first differences were used originally.

Exercises.—Apply Simpson's Rule to the integration of x' from $t=2$ to $t=12$ in the table of section 11.

Check the integrations of x' and y' from $t=12$ to $t=48$.

Check the integrations of Ex' and $Ey'+g$ from $t=2$ to $t=12$ and from $t=12$ to $t=48$.

Check the integrations from $t=0$ to $t=1$ by means of the trapezoid formula.

Check the integrations in the table of section 14.

Exercises.—Compute the entire trajectory corresponding to $C=3.6$, $v_0=2,400$ foot-seconds, $\phi=20°$. Find the time of flight, range, terminal velocity, angle of fall, and maximum ordinate. (The details of the computation should be kept on file, for use in connection with exercises on differential variations.)

Perform the same computation for

$C=2$, $v_0=1,500$ foot-seconds, $\phi=30°$;
$C=5$, $v_0=3,000$ foot-seconds, $\phi=45°$.

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CHAPTER II.

COMPUTATION OF DIFFERENTIAL VARIATIONS.

1. *Importance of a special method for differential variations.*—Suppose that a trajectory has been computed, and that it is required to find the change produced by a variation in the initial velocity, for example, or in the density of the air. An obvious way of getting the result would be to recompute the trajectory, taking account of the changed conditions, and compare the ranges and other quantities found in the two cases. It is found preferable, however, to regard the differences between the coordinates of the projectile on the two trajectories as new unknown functions of t, to write down differential equations which have these functions for solutions, and to treat the new equations, or a system of equations derived from them, by a method of numerical integration similar to that used in the original trajectory computation. There are two principal reasons for this procedure. One is that if two nearly equal quantities are to be compared, errors that are small relatively to the quantities themselves may be large in comparison with their difference, so that the computation of differential variations by means of separate trajectories is inaccurate. The other reason is that the use of the differential equations for the variations is economical of labor, particularly in a systematic determination of the effects of various disturbing influences, such as is necessary in range table work.

A straightforward method of dealing with the problem was laid down by Major Moulton, and his derivation of the equations was somewhat simplified by Lieut. Milne, whose treatment is substantially followed here. A further modification, proposed by Professor Bliss, is of great importance for reducing the labor of computation to a minimum, and is explained later. The methods of Major Moulton and of Professor Bliss, down to a certain point, are described in the next six sections. The former is given in sections 2, 3, 4, 5, the latter in sections 2, 4, 6, 7; either group of four sections may be read independently of the remaining two. A still further improvement in the application of the latter method, due to Dr. Gronwall, is given in section 8.

2. *The fundamental equations for the variations.*—Let x and y represent the coordinates of the projectile on the original trajectory. They satisfy the differential equations

$$x'' = -\mathrm{E}x', \qquad y'' = -\mathrm{E}y' - g. \tag{1}$$

On the modified trajectory, x and y and their derivatives will have new values, at any instant t, and the value of E also will be different, because of a change in H for a given value of y, for example, in case there is such a change, and in any case because of the changes in x', y', and y, on which the values of G and H depend. Let the coordinates on the modified trajectory at the instant t be $x+\xi$ and $y+\eta$, so that ξ and η represent the changes due to the disturbance under consideration; and let $\mathrm{E}+\Delta\mathrm{E}$ be the corresponding value of the retardation function. The new variables will satisfy the equations [1]

$$x''+\xi'' = -(\mathrm{E}+\Delta\mathrm{E})(x'+\xi'),$$
$$y''+\eta'' = -(\mathrm{E}+\Delta\mathrm{E})(y'+\eta') - g.$$

If these are multiplied out, and simplified by means of the relations (1), the following equations are obtained:

$$\xi'' = -\mathrm{E}\xi' - x'\Delta\mathrm{E} - \xi'\Delta\mathrm{E},$$
$$\eta'' = -\mathrm{E}\eta' - y'\Delta\mathrm{E} - \eta'\Delta\mathrm{E}.$$

The differences ξ and η and their derivatives, and also $\Delta\mathrm{E}$, are to be thought of as small quantities, so that the terms $\xi'\Delta\mathrm{E}$ and $\eta'\Delta\mathrm{E}$ are small quantities of the second order. If they are regarded as negligible and left out of account, as may be done with sufficient accuracy for practical purposes, the equations that remain for determining ξ and η are:

$$\xi'' = -\mathrm{E}\xi' - x'\Delta\mathrm{E}, \tag{2}$$
$$\eta'' = -\mathrm{E}\eta' - y'\Delta\mathrm{E}.$$

Since $\Delta\mathrm{E}$ depends on ξ', η' and η, the unknowns are involved implicitly in the second term on the right in each equation, as well as explicitly in the other terms. The quantities x', y', y, and E, on the other hand, are to be thought of as known, since it is assumed that the original trajectory has been computed in advance.

[1] The effect of wind will be taken up later. It is assumed for the present that the air is at rest; otherwise a distinction would have to be made between the velocity of the projectile relatively to the ground, with the components $x'+\xi'$ and $y'+\eta'$, and its velocity relatively to the air, on which the resistance would depend.

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3. *Modified form of the equations.*[1]—In the equations (2) let E be replaced by its value as given by the relation $x'' = -Ex'$, namely $E = -x''/x'$; then

$$\xi'' = \frac{x''\xi'}{x'} - x'\Delta E,$$

$$\eta'' = \frac{x''\eta'}{x'} - y'\Delta E,$$

or, on division by x' and transposition,

$$\frac{x'\xi'' - x''\xi'}{x'^2} = -\Delta E,$$

$$\frac{x'\eta'' - x''\eta'}{x'^2} = -\frac{y'}{x'}\Delta E.$$

The left-hand members here are the derivatives of ξ'/x' and η'/x', respectively. Let x_0' and y_0' stand for the components of the initial velocity on the original trajectory, and let ξ_0' and η_0' be the amounts by which the initial components for the varied trajectory differ from x_0' and y_0'. If there is no change in the direction or magnitude of the initial velocity, the variation being produced altogether by some other departure from normal conditions, ξ_0' and η_0' will be zero. On integration of the last pair of equations from o to t, it is found that

$$\frac{\xi'}{x'} - \frac{\xi_0'}{x_0'} = -\int_0^t \Delta E\, dt,$$

$$\frac{\eta'}{x'} - \frac{\eta_0'}{x_0'} = -\int_0^t \frac{y'}{x'}\Delta E\, dt,$$

that is,

$$\xi' = \frac{x'\xi_0'}{x_0'} - x'\int_0^t \Delta E\, dt, \tag{3}$$

$$\eta' = \frac{x'\eta_0'}{x_0'} - x'\int_0^t \frac{y'}{x'}\Delta E\, dt.$$

Even when the equations are written in this form, however, the process of numerical integration must be carried out step by step, by the principle of successive approximations, because the unknown quantities are involved in the expression ΔE under the integral sign.

4. *Explicit expression for* ΔE.—It is still necessary[2] to know the form of the dependence of ΔE on ξ', η', and η. Since $E = GH/C$, the variation of E will be given approximately in terms of the variations of its factors by the equation

$$\frac{\Delta E}{E} = \frac{\Delta G}{G} + \frac{\Delta H}{H} - \frac{\Delta C}{C}.$$

The changes in G and H will be due, either in part or entirely, to the differences in the values of x', y', and y between the two trajectories.

If y is changed by an amount η, the corresponding change in H will be, to the first order of small quantities,

$$\Delta H = \frac{dH}{dy}\eta, \tag{4}$$

so that

$$\frac{\Delta H}{H} = \frac{1}{H}\cdot\frac{dH}{dy}\cdot\eta = \frac{d\log H}{dy}\eta.$$

When H is given by the exponential formula

$$H = 10^{-.000045y} = e^{-.0001036y},$$

the logarithmic derivative has the constant value

$$\frac{d\log H}{dy} = -.0001036.$$

[1] This section is not needed for the discussion of the later methods. The transformation given here is also not necessary for the direct integration of the equations (2), but it has been found convenient in practice, and the resulting formulas are used in section 5.

[2] This applies equally to all methods.

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Let Δv be the change in v corresponding to changes of ξ' and η' in x' and y', respectively. If small quantities of order higher than the first are left out of account,

$$\Delta v = \Delta\sqrt{x'^2+y'^2} = \frac{\Delta(x'^2+y'^2)}{2\sqrt{x'^2+y'^2}} = \frac{2x'\xi'+2y'\eta'}{2\sqrt{x'^2+y'^2}} = \frac{1}{v}(x'\xi'+y'\,\eta').$$

Consequently

$$\Delta G = \frac{dG}{dv}\Delta v = \frac{1}{v}\frac{dG}{dv}(x'\xi'+y'\eta'), \tag{5}$$

and

$$\frac{\Delta G}{G} = \frac{1}{vG}\frac{dG}{dv}(x'\xi'+y'\eta') = \frac{1}{v}\frac{d\log G}{dv}(x'\xi'+y'\eta').$$

There is no simple expression for $d \log G/dv$, but its values are tabulated, like those of G, with $v^2/100$ as argument.

Let δE stand for the part of the change ΔE that is due to the variations in x', y', and y. Then

$$\delta E = E\left[\frac{1}{v}\frac{d\log G}{dv}(x'\xi'+y'\eta') + \frac{d\log H}{dy}\eta\right].$$

It is easy now to calculate the total change ΔE corresponding to various cases of disturbance.

For a *change in the initial components of velocity*, without any other departure from normal conditions,

$$\Delta E = \delta E.$$

For a *change in ballistic coefficient*,

$$\Delta E = \delta E - E\frac{\Delta C}{C}. \tag{6}$$

For a *change in air density*, the result can be obtained by regarding the variation in density as equivalent to an inverse change in the value of the ballistic coefficient, provided that the percentage change in density is the same at all altitudes. This is, of course, a very special case, but it is made the basis for the treatment of general density variations in practice, as is explained in section 9.

The above, together with the effect of a rear or head wind, to be taken up in the next section, are the range variations that are of most importance. There is no difficulty, however, in writing down the value of ΔE in other cases. Consider, for example, a change in density which is not the same proportionally at different heights. Let $H(y)$ on the varied trajectory be taken to mean the ratio of the density at height y to the surface density that is assumed as normal. Let ΔH stand for the total difference in H between the two trajectories at any instant t, δH for the part of ΔH that is due to the change in y, and $\delta_0 H$ for the part of ΔH that is due to the change in the value of H for a fixed value of y. Then δH is the same as the quantity that was denoted by ΔH in (4). It would have been more consistent to use the symbol δH in the first place, but it was preferred not to complicate the notation in the important cases where δH and ΔH are the same. In formulas,

$$\Delta H = \delta H + \delta_0 H,$$

and

$$\Delta E = \delta E + E\frac{\delta_0 H}{H}.$$

The factor $\delta_0 H/H$ will vary along the trajectory, in general. If it is constant, the formula just obtained is equivalent to (6), since an increase in H corresponds to the same percentage decrease in C, to terms of the first order.

If there is a change in the form of the retardation function G, let

$$\Delta G = \delta G + \delta_0 G,$$

where δG is the same as ΔG in (5), and $\delta_0 G$ is the change in G for a fixed v. Then

$$\Delta E = \delta E + E\frac{\delta_0 G}{G}.$$

It is to be remembered that the equations obtained in this section are not exact, but are merely correct to the first order of small quantities.

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Exercise.—Verify the tabulated values of $d \log G/(vdv)$ approximately, for $v^2/100 = 1{,}000, 2{,}000, 3{,}000$, by taking differences in the G-table, and calculating the values of $\Delta \log G/(v\Delta v)$, noticing that the G-table gives logarithms to base 10, while the logarithm to be differentiated is the natural logarithm, and that $v\Delta v$ is approximately $\frac{1}{2}\Delta(v^2)$.

5. *Variation due to rear or head wind.*[1]—Suppose now that the air is not at rest, but that there is a rear wind, of constant velocity u. The treatment of a variable wind, in a later section, will be made to depend on this special case. A head wind would correspond to a negative value of u. It would be possible to modify the equations (2) so as to take account of the effect of the wind, and this will be done in one of the following sections. The result can also be obtained, however, by making appropriate changes in the equations (3) directly. The motion of the projectile *relative to the air* is the same as if no wind were blowing, and the horizontal component of the velocity of the projectile were diminished by u. If the components of velocity relative to the ground on the varied trajectory are $x'+\xi'$ and $y'+\eta'$, the components relative to the air will be $x'+\xi'-u$ and $y'+\eta'$. In (3), ξ' has to be replaced by $\xi'-u$, both in the left-hand member of the first equation and in the explicit form of ΔE from section 4; and since $\xi_0'=0$, if the wind effect is the only one under consideration, the initial value of $\xi'-u$ is $-u$. The equations become

$$\xi'-u=-\frac{x'u}{x_0'}-x'\int_0^t \Delta E\,dt,$$

$$\eta'=-x'\int_0^t \frac{y'}{x'}\Delta E\,dt,$$

where

$$\Delta E=E\left[\frac{1}{v}\frac{d\log G}{dv}(x'\xi'+y'\eta'-x'u)+\frac{d\log H}{dy}\eta\right],$$

so that ξ and η themselves are to be found from the relations

$$\xi'=u\left(1-\frac{x'}{x_0'}\right)-x'\int_0^t \Delta E\,dt,$$

$$\eta'=-x'\int_0^t \frac{y'}{x'}\Delta E\,dt,$$

with the value of ΔE just given.

Exercises.—Taking as standard the trajectory for

$$C=3.6, \qquad v_0=2{,}400 \text{ foot-seconds}, \qquad \phi=20^\circ,$$

compute the change in range due to:

A change of 1 meter per second in muzzle velocity;

A change of 1 per cent in ballistic coefficient;

A rear wind of 1 meter per second.

6. *The adjoint system of differential equations.*—When the value of ΔE from section 4 is substituted in the equations (2), for any one of the variations, it is found that the equations are of the first degree in ξ'', η'', ξ', η', and η; the coefficients are not constants, or at any rate most of them are not, but they are known functions of t, when the original trajectory has been computed. It is convenient to use new letters[2] for ξ' and η', say σ and τ. If

(7)
$$\xi'=\sigma,\ \eta'=\tau,$$

then

$$\xi''=\sigma',\ \eta''=\tau',$$

and it can be considered that the equations (2) give σ' and τ' as expressions of the first degree in σ, τ, and η. Moreover, the equations (7) have a similar form, to the extent that they give ξ' and η' as very simple expressions of the first degree in σ and τ. The purpose of these remarks, trivial enough in themselves, is to bring out the fact that the four equations (2) and (7) can be regarded as coming under the general form

(8)
$$\begin{aligned}\xi'&=a_1\xi+b_1\eta+c_1\sigma+d_1\tau+e_1,\\ \eta'&=a_2\xi+b_2\eta+c_2\sigma+d_2\tau+e_2,\\ \sigma'&=a_3\xi+b_3\eta+c_3\sigma+d_3\tau+e_3,\\ \tau'&=a_4\xi+b_4\eta+c_4\sigma+d_4\tau+e_4,\end{aligned}$$

[1] This section is not needed for the discussion of the later methods, as the question of wind is taken up again from the start in section 7.

[2] The letter σ used here has no connection with the same letter as used in the tangent-reciprocal method of computing trajectories.

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a system of the first order and the first degree in four unknown quantities. Ten or more of the 20 coefficients on the right will always be zero in the present problem, but the reasons for the steps that are to be taken will be more apparent if the general expressions are used. In the next section the values of the coefficients will be written down explicitly; it will be shown that the equations can be put into the form (8), not only in the case of the variations that are discussed in section 4, but also in the case of a rear or head wind; and the advantages of the present method of dealing with the problem will be indicated. The rest of the present section will be given up to the derivation of a general formula relating to systems of differential equations of the form (8).

Let a new system of differential equations, containing four new unknown quantities λ, μ, ν, ρ, be written down as follows:

$$\begin{aligned} -\lambda' &= a_1\lambda + a_2\mu + a_3\nu + a_4\rho, \\ -\mu' &= b_1\lambda + b_2\mu + b_3\nu + b_4\rho, \\ -\nu' &= c_1\lambda + c_2\mu + c_3\nu + c_4\rho, \\ -\rho' &= d_1\lambda + d_2\mu + d_3\nu + d_4\rho. \end{aligned} \tag{9}$$

The 16 coefficients on the right are the same as the 16 coefficients of ξ, η, σ, and τ in (8), but differently arranged, rows in one case corresponding to columns in the other. There are no terms corresponding to $e_1, \ldots, e_4$, and the left-hand members have negative signs.[1] The new system is called *adjoint* to the system (8).

Suppose that four functions λ, μ, ν, ρ have been found to satisfy the equations (9). Let the four equations of (8) be multiplied by these four functions, respectively, and added. On the left, the expression

$$\lambda\xi' + \mu\eta' + \nu\sigma' + \rho\tau'$$

will be obtained. On the right, the coefficient of ξ will be $a_1\lambda + a_2\mu + a_3\nu + a_4\rho$; but this, by the first equation of (9), is the same as $-\lambda'$. The coefficients of η, σ, and τ will be simplified in the same way. The resulting equation is

$$\begin{aligned} \lambda\xi' + \mu\eta' + \nu\sigma' + \rho\tau' &= -\lambda'\xi - \mu'\eta - \nu'\sigma - \rho'\tau \\ &\quad + \lambda e_1 + \mu e_2 + \nu e_3 + \rho e_4. \end{aligned} \tag{10}$$

Let the terms with negative signs be transposed to the left-hand side. The combination $\lambda\xi' + \lambda'\xi$ is the derivative of the product $\lambda\xi$, and similarly for each of the other three pairs of corresponding terms. Because of this, (10) becomes

$$(\lambda\xi + \mu\eta + \nu\sigma + \rho\tau)' = \lambda e_1 + \mu e_2 + \nu e_3 + \rho e_4,$$

or, by integration from $t=0$ to $t=T$,

$$[\lambda\xi + \mu\eta + \nu\sigma + \rho\tau]_{t=0}^{t=T} = \int_0^T (\lambda e_1 + \mu e_2 + \nu e_3 + \rho e_4)\,dt. \tag{11}$$

7. *Application of the adjoint system to the problem of differential variations.*—It has been pointed out that if any one of the values of ΔE from section 4 is substituted in the equations (2), and if ξ' and η' are represented by σ and τ, a set of four differential equations of the form (8) is obtained for the four unknown quantities ξ, η, σ, and τ. The actual result of the substitution, in the case of a variation due to a change in the initial velocity components, is as follows:

$$\begin{aligned} &\xi' = \sigma, \quad \eta' = \tau, \\ &\sigma' = -Ex'\frac{d\log H}{dy}\eta - E\left(1 + \frac{x'^2}{v}\frac{d\log G}{dv}\right)\sigma - \frac{Ex'y'}{v}\frac{d\log G}{dv}\tau, \\ &\tau' = -Ey'\frac{d\log H}{dy}\eta - \frac{E\,x'y'}{v}\frac{d\log G}{dv}\sigma - E\left(1 + \frac{y'^2}{v}\frac{d\log G}{dv}\right)\tau. \end{aligned} \tag{12}$$

That is, $c_1 = 1$, $d_2 = 1$, $b_3 = -Ex'(d\log H)/dy$, etc., while a large number of the coefficients, including all the e's, are zero. For the other variations considered in section 4, since the only difference from the case just treated is the introduction of a term in ΔE which does not involve ξ', η', or η, there will be no change in the resulting equations, except that e_3 and e_4 will have values different from zero. For a variation in ballistic coefficient,

$$e_3 = Ex'\frac{\Delta C}{C}, \qquad e_4 = Ey'\frac{\Delta C}{C};$$

for a change in the form of H,

$$e_3 = -Ex'\frac{\delta_0 H}{H}, \qquad e_4 = -Ey'\frac{\delta_0 H}{H};$$

[1] In the ballistic problem, although σ and τ are simply the derivatives of ξ and η, there will be no such simple relation connecting λ, μ, ν, and ρ.

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and for a corresponding change in G,

$$e_3 = -Ex'\frac{\delta_0 G}{G}, \qquad e_4 = -Ey'\frac{\delta_0 G}{G}.$$

Equations of similar form are found to hold for the variation due to a rear wind. Let the velocity of the wind be u. The original trajectory is given by the equations (1). On the varied trajectory the components of the retardation are determined by the motion of the projectile relative to the air, and are the same as if the air were at rest and the horizontal component of the velocity of the projectile, $x'+\xi'$, were diminished by u. That is, the equations of motion for the varied trajectory are

$$\begin{aligned} x''+\xi'' &= -(E+\Delta E)\,(x'+\xi'-u), \\ y''+\eta'' &= -(E+\Delta E)\,(y'+\eta')-g, \end{aligned} \tag{13}$$

where ΔE stands for the change in E produced by changes of $\xi'-u$, η', and η, in x', y', and y. The equations corresponding to (2), obtained by combining (1) and (13) and neglecting small terms of order higher that the first, are

$$\begin{aligned} \xi'' &= -E(\xi'-u)-x'\Delta E, \\ \eta'' &= -E\eta'-y'\Delta E. \end{aligned} \tag{14}$$

The value of ΔE is the expression δE of section 4, with ξ' replaced by $\xi'-u$:

$$\Delta E = E\left[\frac{1}{v}\frac{d\log G}{dv}(x'\xi'+y'\eta'-x'u)+\frac{d\log H}{dy}\eta\right].$$

Substitution of this in (14) leads to equations of the form (8), with the same coefficients as before for η, σ, and τ, and with

$$e_3 = Eu\left(1+\frac{x'^2}{v}\frac{d\log G}{dv}\right), \qquad e_4 = Eu\frac{x'y'}{v}\frac{d\log G}{dv}.$$

All the variations, then, are given by equations of the form (8), with exactly the same coefficients in each case, except for the terms e_3 and e_4. The adjoint system (9) does not involve these terms and is therefore the same for all the variations. It will be shown later in the section how the effect of the disturbance in question on the range can be found at once when the adjoint system has been solved. The advantage in the use of the adjoint system then is that there is just one set of differential equations to be solved by a process of numerical integration, instead of a number of different sets, one for each variation. The advantage is much greater even than appears at first sight. For it will be seen in the second section following that in order to make allowance for a variable wind, for example, while the problem is made to depend on the case of a constant wind, it is necessary to know not only the effect of a constant wind blowing throughout the whole trajectory, but also the effect of a wind blowing only a part of the time, and this for a sufficient number of different parts. When the adjoint system is not used each part calls for a separate solution by numerical integration, while one solution of the adjoint system, with one set of initial conditions, accomplishes the whole purpose. In behalf of the more direct method it is to be said that it is somewhat easier to follow, in practice as well as in theory, because of the more obvious physical significance of the quantities involved. This may be worth taking into account if a single variation is wanted for a particular purpose, rather than the whole set of variations. Moreover, it gives directly the values of ξ, η, ξ', and η', all along the trajectory, which is sometimes an advantage.

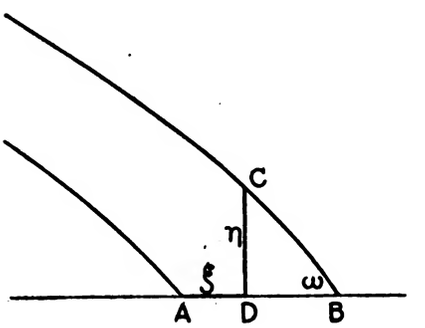

Fig. 3

To see how the variation in range is to be obtained from the solution of the adjoint system, let A, in the figure, be the point of fall on the original trajectory, and B the point of fall on the varied trajectory, so that AB is the distance to be determined. Let C be the position of the projectile on the varied trajectory at the instant $t=T$, when it would have been at A on the original path. The distance AD is ξ, and DC is η, for $t=T$. If ω stands for the angle CBD, then

$$\Delta X = AB = \xi+\eta\cot\omega.$$

It is sufficiently accurate to regard ω as the angle of fall on the original trajectory, for, as cot ω is multiplied by the small quantity η, a small error in the angle affects the value found for AB only by a small quantity of the second order.

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Now, suppose that the adjoint system is integrated, not with given values of the unknown functions for $t=0$, but with the values

$$\lambda=1, \qquad \mu=\cot\omega, \qquad \nu=0, \qquad \rho=0,$$

for $t=T$, the actual numerical calculation being carried backwards from $t=T$ to $t=0$. Then the desired variation in range is given by the formula (11) at the end of the preceding section. For, $\xi+\eta\cot\omega$ is the value of the bracket for $t=T$, and the formula states that

$$\xi+\eta\cot\omega=[\lambda\xi+\mu\eta+\nu\sigma+\rho\tau]_{t=0}+\int_0^T(\lambda e_1+\mu e_2+\nu e_3+\rho e_4)dt. \tag{15}$$

The quantities that appear on the right are either known at the start or given by the solution. It remains to evaluate the integral, but this can be done by direct summation, and does not involve any new process of successive approximations.

The form of the result can be made more explicit for the particular variations that have to be considered. It has been emphasized already that the adjoint system of differential equations is the same in all cases. Written out in detail, the equations are as follows:

(16)

$$\lambda'=0,$$

$$\mu'=Ex'\frac{d\log H}{dy}\nu+Ey'\frac{d\log H}{dy}\rho=E\frac{d\log H}{dy}(x'\nu+y'\rho),$$

$$\nu'=-\lambda+E\left(1+\frac{x'^2}{v}\frac{d\log G}{dv}\right)\nu+E\frac{x'y'}{v}\frac{d\log G}{dv}\rho$$

$$=-\lambda+E\nu+\frac{Ex'}{v}\frac{d\log G}{dv}(x'\nu+y'\rho),$$

$$\rho'=-\mu+E\frac{x'y'}{v}\frac{d\log G}{dv}\nu+E\left(1+\frac{y'^2}{v}\frac{d\log G}{dv}\right)\rho$$

$$=-\mu+E\rho+\frac{Ey'}{v}\frac{d\log G}{dv}(x'\nu+y'\rho).$$

In particular, the first equation shows that λ is a constant, and since $\lambda=1$ for $t=T$, it follows that $\lambda=1$ throughout.

For a change in the components of the initial velocity, without any other disturbance, all the e's are zero. Moreover, ξ and η are zero for $t=0$, while ξ_0' and η_0' are given. Let the values of ν and ρ for $t=0$, found from the solution of the differential equations, be ν_0 and ρ_0. Then

$$\Delta X=\nu_0\,\xi_0'+\rho_0\,\eta_0'.$$

For a change in ballistic coefficient, taken by itself, all four of the quantities ξ, η, σ, and τ are zero for $t=0$. On substitution of the values of e_3 and e_4 given earlier in the section, it is found that

$$\Delta X=\frac{\Delta C}{C}\int_0^T E(x'\nu+y'\rho)dt.$$

In the case of a rear wind, the parenthesis under the sign of integration in (15) is equal to

$$u\left[E\nu+\frac{Ex'}{v}\frac{d\log G}{dv}(x'\nu+y'\rho)\right].$$

This can be much simplified, however, by the use of the third equation of the adjoint system, from which it appears that the quantity in brackets is equal to $\nu'+\lambda$, or, since $\lambda=1$, to $\nu'+1$. Consequently, in this case,

$$\Delta X=\int_0^T u(\nu'+1)dt. \tag{17}$$

In practice, u will be taken as constant, and can be placed as a factor before the sign of integration; the formula as written, however, holds equally well for variable u.

The account that has been given is to be regarded as only an introduction to the use of the adjoint system, even in the case of the variations that have been treated explicitly. The form of the equations can be modified in various ways, with a greater or less saving of labor in computation (see particularly section 8), and their use is not by any

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means restricted to the determination of the single quantity ΔX. For example, it is possible to find the quantities ξ, η, ξ', and η' separately, or any linear combination of them, by properly choosing the values of λ, μ, ν, and ρ for $t=T$. In general, it may be said that the adjoint system can be used to advantage whenever it is desired to find the changes in any one element of a trajectory produced by a variety of causes.

Exercises.—Compute the differential variations named at the end of section 5, using the adjoint system. (For an improved method of computation, however, see the following section.)

8. *First integrals of the adjoint system.*—The work of computation can be considerably reduced by a method proposed by Dr. Gronwall, which depends on the discovery of what is known as a first integral of the adjoint system.

The fundamental differential equations for the variations can be written in the form:

$$\xi'' = -Ex'\frac{d\log H}{dy}\eta - E\left(1+\frac{x'^2}{v}\frac{d\log G}{dv}\right)\xi' - E\frac{x'y'}{v}\frac{d\log G}{dv}\eta', \tag{18}$$

$$\eta'' = -Ey'\frac{d\log H}{dy}\eta - E\frac{x'y'}{v}\frac{d\log G}{dv}\xi' - E\left(1+\frac{y'^2}{v}\frac{d\log G}{dv}\right)\eta'.$$

These two equations of the second order are equivalent to the four equations of the first order, numbered (12), at the beginning of section 7. The variable ξ, as distinguished from its derivatives, does not appear in these equations; and it has already been seen that, as a consequence, the first equation of the adjoint system reduces to $\lambda'=0$, which can be integrated directly, giving $\lambda=$ constant. This relation is called a first integral of the adjoint system; it leaves only three equations, instead of four, to be treated at length by numerical integration.

Dr. Gronwall has found another general integral, the existence of which can be shown to depend on the fact that the equations (18) do not contain t explicitly. It will be sufficient for present purposes to verify the formula by a more direct method.

Let the equations (1) be differentiated with regard to t. The result is:

$$x''' = -Ex'' - x'\frac{dE}{dt}, \qquad y''' = -Ey'' - y'\frac{dE}{dt}. \tag{19}$$

Since E has the form G (v) H (y)/C, in which v and y vary with t, while C is constant:

$$\frac{1}{E}\frac{dE}{dt} = \frac{1}{G}\frac{dG}{dt} + \frac{1}{H}\frac{dH}{dt}$$

$$= \frac{1}{G}\frac{dG}{dv}\frac{dv}{dt} + \frac{1}{H}\frac{dH}{dy}\frac{dy}{dt}$$

$$= \frac{d\log G}{dv}\frac{dv}{dt} + \frac{d\log H}{dy}y'.$$

Furthermore,

$$v=\sqrt{x'^2+y'^2}, \quad \frac{dv}{dt}=\frac{x'x''+y'y''}{\sqrt{x'^2+y'^2}}=\frac{1}{v}(x'x''+y'y''),$$

and so

$$\frac{dE}{dt} = E\left[\frac{1}{v}\frac{d\log G}{dv}(x'x''+y'y'') + \frac{d\log H}{dy}y'\right].$$

The resemblance of this to the formula for δE in section 4 is apparent. If the expanded form of dE/dt is substituted in (19), the terms can be collected as follows:

$$x''' = -Ex'\frac{d\log H}{dy}y' - E\left(1+\frac{x'^2}{v}\frac{d\log G}{dv}\right)x'' - E\frac{x'y'}{v}\frac{d\log G}{dv}y'', \tag{20}$$

$$y''' = -Ey'\frac{d\log H}{dy}y' - E\frac{x'y'}{v}\frac{d\log G}{dv}x'' - E\left(1+\frac{y'^2}{v}\frac{d\log G}{dv}\right)y''.$$

These are exactly the same as the equations that would be obtained if x' and y' were substituted for ξ and η in (18). In other words, since the result of the substitution is the pair of equations (20), which have just been proved to be true, it follows that *the functions x' and y' constitute a particular solution of the differential equations* (18). They do not give an immediate solution of the whole problem of differential variations, even in the simplest cases, because they do not satisfy the right initial conditions; but they do have an application in shortening the work.

For the equations (12) at the beginning of section 7, a set of solutions is given by

$$\xi=x', \qquad \eta=y', \qquad \sigma=x'', \qquad \tau=y''.$$

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The equations (12) are the same as those numbered (8) in section 6, with the general coefficients replaced by their particular values. It was seen at the end of section 6 that if ξ, η, σ, τ are *any* functions satisfying these equations, and λ, μ, ν, ρ are any functions satisfying the adjoint system, then

$$(\lambda\xi+\mu\eta+\nu\sigma+\rho\tau)'=\lambda e_1+\mu e_2+\nu e_3+\rho e_4.$$

Since, in the present case, the e's are all zero, this means that

$$\begin{aligned}(\lambda x'+\mu y'+\nu x''+\rho y'')'&=0,\\ \lambda x'+\mu y'+\nu x''+\rho y''&=\text{constant}.\end{aligned}$$

Let the value of this constant be denoted by $-k$; it can be determined, before beginning the integration of the adjoint system, from the known values of $\lambda, \mu, \nu, \rho, x', y', x'$, and y'', for $t=T$, or for the instant at which the corrections are to be computed, whatever that may be. In determining range corrections, as was explained in section 7, the values to be used for the end of the trajectory are $\lambda=1, \mu=\cot\omega, \nu=0, \rho=0$; and as $x'=-y'\cot\omega$ for $t=T$, it follows that in this case $k=0$. There are other applications of the adjoint system, however, in which k would have a different value. The relation

$$k+\lambda x'+\mu y'+\nu x''+\rho y''=0 \tag{21}$$

can be used to reduce the number of equations for numerical integration from three to two.

It will be convenient to make use of the fact that for practical purposes the quantity $d\log H/dy$, which appears in the differential equations, is a constant. Let this constant be denoted by $-h$. Two functions μ_a, μ_b, will be defined by the relations

$$\mu_a=\frac{\mu}{h}, \qquad \mu_b=\mu'_a;$$

and the problem will be reduced to that of solving a pair of equations for μ_a and μ_b. The constant value of λ will be denoted by l. It has been seen that for the problem of range corrections $l=1$.

With the notation just introduced, the equations (16) can be written in the form

$$\begin{aligned}\lambda'&=0,\\ \mu'_a&=-E(x'\nu+y'\rho),\\ \nu'&=-l+E\nu+\frac{Ex'}{v}\frac{d\log G}{dv}(x'\nu+y'\rho),\\ \rho'&=-h\mu_a+E\rho+\frac{Ey'}{v}\frac{d\log G}{dv}(x'\nu+y'\rho).\end{aligned} \tag{22}$$

Let x'' and y'' in (21) be replaced by their values from (1); then (21) becomes

$$k+lx'+h\mu_a y'-E(x'\nu+y'\rho)-g\rho=0,$$

and this, by the second equation of (22) and the definition of μ_b, is the same as

$$k+lx'+h\mu_a y'+\mu_b=g\rho. \tag{23}$$

From the second and fourth of equations (22),

$$\rho'=-h\mu_a+E\rho-\frac{y'}{v}\frac{d\log G}{dv}\mu_b. \tag{24}$$

By differentiation of (23),

$$\begin{aligned}g\rho'&=lx''+h\mu_a'y'+h\mu_a y''+\mu_b'\\ &=-lEx'+h\mu_b y'-h\mu_a Ey'-h\mu_a g+\mu_b'.\end{aligned}$$

With this value for $g\rho'$, and the expression (23) for $g\rho$, the relation (24), multiplied by g, takes the form

$$-lEx'+h\mu_b y'-h\mu_a Ey'-h\mu_a g+\mu'_b$$

$$=-h\mu_a g+E(k+lx'+h\mu_a y'+\mu_b)-\frac{gy'}{v}\frac{d\log G}{dv}\mu_b,$$

or

$$\mu'_b=2hEy'\mu_a+\left(E-hy'-\frac{gy'}{v}\frac{d\log G}{dv}\right)\mu_b+E(k+2lx').$$

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This, together with the relation

$$\mu'_a = \mu_b,$$

forms the pair of equations for μ_a and μ_b to which the solution of the original problem was to be reduced. The required functions λ, μ, ν, ρ, are expressed in terms of μ_a, μ_b, and the known constants k and l, as follows:

$$\lambda = l,$$
$$\mu = h\mu_a,$$
$$\rho = \frac{1}{g}(k + lx' + hy'\mu_a + \mu_b),$$

and, from the second equation of (22)

$$\nu = -\frac{\mu_b}{Ex'} - \frac{y'\rho}{x'}.$$

Exercises.—Compute the differential variations named at the end of section 5, using the method of the present section.

9. *Variable wind and air density—Weighting factors.*—On any given occasion the wind will not be constant throughout the flight of a projectile, but will vary considerably in direction and velocity at different heights above the ground. For practical purposes it is assumed that the wind at the time of firing depends only on the height, and that its direction and intensity are known from meteorological observations at different altitudes. In the present section only the component in the plane of fire is to be considered, the cross component being left for later treatment. The observations are not such as to justify any great refinement in the calculations based on them, and it is sufficiently accurate to use an average value of the wind for a certain height above the earth's surface, 250 meters, for example, another average value for the next zone of 250 meters, and so on until the maximum ordinate is reached. The manner of finding the combined effect of these different winds is as follows:

For the trajectory in question, let ΔX_0 be the range effect of a uniform wind of given intensity, say 1 meter per second, blowing throughout the trajectory. If y_1 is the height selected for the first zone, let ΔX_1 be the range effect which would be produced if a unit wind were blowing in this zone, and the air above the height y_1 were at rest. If the computation is performed by means of the adjoint system, ΔX_1 is obtained from the formula (17) simply by extending the integral from o to the instant when the projectile on the original trajectory[1] leaves the zone, and from the instant when it reenters the zone to T, and leaving out the intermediate part. If the equations (2) are integrated directly, the effect of the wind is computed up to the instant when the projectile leaves the zone, then the effect produced on its further flight by the changes in x', y', and y already found, and finally the combined effect of the wind and the previous changes from the time when the projectile enters the zone again until it reaches the ground. Similarly, let ΔX_2 be the effect of a unit wind which blows only in the second height zone, and so on as far as it may be necessary to go. If there are n zones,

$$\Delta X_0 = \Delta X_1 + \Delta X_2 + \ldots + \Delta X_n.$$

In terms of the quantities just introduced, it is easy to express the total effect of any wind that varies from zone to zone. Let u_1 be the average velocity in the first zone, u_2 the average in the second zone, and so on. Then the total range effect is

$$\Delta X = u_1 \Delta X_1 + u_2 \Delta X_2 + \ldots + u_n \Delta X_n. \tag{25}$$

If there is a rear wind in some zones and a head wind in others, the terms must be taken with the proper algebraic signs. It should be mentioned that the process of combining wind effects by simple addition is not one for which strict theoretical accuracy can be claimed, but is a practical approximation of the same nature as that which is involved in adding the changes produced by different kinds of disturbing influences.

It has been found convenient to use instead of the quantities ΔX_1, ΔX_2, . . ., the ratios obtained by dividing them by ΔX_0. These ratios,

$$f_1 = \frac{\Delta X_1}{\Delta X_0}, \quad f_2 = \frac{\Delta X_2}{\Delta X_0}, \quad \ldots, \quad f_n = \frac{\Delta X_n}{\Delta X_0},$$

[1] More exactly, the time should be that on the modified trajectory, but the error committed in substituting one for the other is of the order of those which are neglected in all this work. It is more convenient to use the time taken from the original trajectory computation.

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are called *weighting factors*. They tell what part of the total effect of a uniform wind is due to its action in the respective zones; they indicate the relative importance of the various zones in determining the resultant range change. The sum of all the weighting factors is unity. By means of them, equation (25) can be written in the form

$$\Delta X = (f_1 u_1 + f_2 u_2 + \cdots + f_n u_n)\Delta X_0.$$

The factor

$$f_1 u_1 + f_2 u_2 + \cdots + f_n u_n$$

is called the *ballistic range wind*. It is the strength of a uniform wind which would produce the same effect as the variable wind actually observed. To speak more exactly, in recognition of the inaccuracies in the methods of calculation employed, the last statement is the definition of the ballistic wind, and the formula gives an approximation to it. As in the case of other methods that have been discussed, the definition and manner of use of the weighting factors can be modified in various ways, and the form presented here is chosen merely to illustrate what is essential in the idea.

On the same principle, *air density weighting factors* and a *ballistic air density* are introduced in making allowance for variations from normal density which are not proportionately the same at different heights. In the first place, the range variation is calculated for a constant percentage change in density which amounts to ten milligrams change in the weight of a liter of air,[1] say, at the earth's surface, the actual change in milligrams per liter at different heights being correspondingly less. Then the partial variations are calculated which would be produced if the same departure from normal density were effective in just one altitude zone. The ratios of these partial changes to the total change are the weighting factors. The principal difference between their use and that of the wind weighting factors is that before they are multiplied in, the observed density variations at different heights have to be reduced to the corresponding variations in surface density. These equivalent surface variations are multiplied by the appropriate weighting factors, and added to give the ballistic density change.

10. *Cross wind.*—In making a range table, it is necessary to consider not only disturbances acting in the plane of fire and affecting the range, but also those which act across the plane of fire and produce a deflection to the right or to the left. One of the most important causes of deflection is that which produces drift. Its mathematical treatment involves the dynamics of the gyroscopic motion of the rotating projectile, and is of quite a different order of difficulty from the questions taken up here, though some progress has been made in the study of it. In the present state of ballistic science, drift is not calculated theoretically, but is determined by experimental firings. The deflection that calls for most attention in computing range tables is that due to cross wind.

Let w be the magnitude of the wind component across the plane of fire, regarded as constant. Let z be the distance of the projectile from the plane of fire at any instant, so that x, y, and z form a system of rectangular coordinates in three dimensions, with the y-axis vertical. It would be a little more consistent with previous notation to regard z as pertaining to the original trajectory, undisturbed by drift or anything else, and so as always equal to zero, the coordinate on the disturbed trajectory being represented by a Greek letter, ζ; but the use of z for the latter trajectory will not cause any confusion. It is customary to regard z as measured positively toward the right, since that is the direction toward which drift takes place in the case of guns with right-handed rifling. In the same way, w will be regarded as positive for a wind blowing toward the right, and negative for a wind blowing toward the left.

The differential equation for the z-coordinate is

$$z'' = -(R + \Delta R)\cos\gamma,$$

where γ is the angle which the direction of motion of the projectile *relative to the air* at any instant makes with the z-axis, and $R + \Delta R$ is the magnitude of the total retardation on the disturbed trajectory. The z-component of the velocity relative to the air is $z' - w$, and $\cos\gamma$ may be represented by $(z' - w)/(v + \Delta v)$. So the equation becomes

$$z'' = -(R + \Delta R)\frac{z' - w}{v + \Delta v} = -(E + \Delta E)(z' - w).$$

Since z' and w are to be regarded as small quantities, however, the terms containing ΔE will be of the second order, and it is sufficient to use the equation

$$z'' = -E(z' - w),$$

where E has the value belonging to the original trajectory, supposed to have been already computed.

[1] The normal weight of a liter of air is taken as 1,203.4 milligrams.

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Let the value of E be substituted from the relation $x'' = -Ex'$, so that

$$z'' = \frac{x''}{x'}(z' - w),$$

or, by clearing of fractions and transposing,

$$z''\,x' - x''\,z' = -wx''.$$

If the last equation is divided by x'^2 on both sides, the derivative of z'/x' is obtained on the left, and the derivative of w/x' on the right. Hence, on integration from o to an arbitrary instant t,

$$\frac{z'}{x'} = \frac{w}{x'} - \frac{w}{x'_0},$$

$$z' = w\left(1 - \frac{x'}{x'_0}\right).$$

A second integration gives

$$z = w\left(t - \frac{x}{x'_0}\right).$$

The total effect, to the time when the projectile reaches the ground, is given by the formula

$$w\mathrm{T} - \frac{w}{x'_0}\mathrm{X}.$$

It is to be noticed that this is exactly what is obtained by considering that the initial motion of the projectile relative to the air is inclined toward the left, for positive w, at an angle whose sine or tangent (not to distinguish between these ratios for a small angle) is w/x'_0, so that the lateral deviation relative to the air in a distance X amounts to $-w\mathrm{X}/x'_0$, while the motion of the air relative to the ground during the flight is $w\mathrm{T}$.

The effect of a variable wind is calculated by means of weighting factors, as in other cases. If t_1 is any instant intermediate between o and T, the deviation caused by a wind of constant strength w blowing from $t = t_1$ to $t = \mathrm{T}$ is

$$w\left(\mathrm{T} - t_1 - \frac{\mathrm{X} - x_1}{x'_1}\right),$$

the quantities x_1 and x'_1 referring to the instant $t = t_1$. The deviation for a wind blowing from $t = t_1$ to $t = t_2$ is found by performing the calculation for a wind effective from $t = t_1$ to $t = \mathrm{T}$ and for one effective from $t = t_2$ to $t = \mathrm{T}$, and taking the difference of the results.

In connection with the method of the last paragraph but one, it is found by writing down the formulas that the resultant initial velocity relative to the air differs only by a small quantity of the second order from the resultant initial velocity relative to the ground, and it can be shown then by a brief process of reasoning that a cross wind component has no first order effect on the range nor on the values of x', y', x, and y at any time, a fact which would readily be believed without proof.

Exercise.—Compute the deviation caused by a cross wind of 1 meter per second in the case of the trajectory for

$$\mathrm{C} = 3.6,\ v_0 = 2{,}400 \text{ foot-seconds},\ \phi = 20°.$$

11. *Other differential variations—Second order terms.*—In addition to the differential variations which have been discussed, there are others which are sometimes of considerable importance, and which can be treated by similar methods. Among these may be mentioned the variations due to the rotation of the earth, to vertical wind, and to the influence of changes in temperature on the form of the function $\mathrm{G}(v)$.

Of less importance is the question of second order variations. Some mention has already been made of the practical assumption that the variations due to different causes can be combined as if they were independent of each other, and of the fact that from a theoretical point of view this is only an approximation to the truth. When a number of disturbing causes act together, the effect of each is modified somewhat by the presence of the others. Such secondary effects have been left out of account, together with other minor sources of inaccuracy.

Some of these points have been studied carefully in connection with the making of range tables. Others can be neglected until improvements have been made in the methods of observation. None of them will be taken up further here, but it is to be said once more that the principle of numerical integration gives a means of attacking each problem as soon as the need for it becomes apparent.

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TABLE I.—G(v).

The table gives $\log_{10}G$, the argument being $v^2/100$, where v is expressed in meters per second.

$v^2/100$	0	1	2	3	4	5	6	7	8	9
0	.....	7.2395	3872	4731	5336	5804	6186	6507	6783	7027
10	7.7244	7440	7619	7782	7934	8074	8205	8328	8443	8552
20	7.8655	8753	8846	8935	9020	9101	9179	9254	9326	9395
30	7.9462	9527	9589	9649	9708	9765	9820	9874	9926	9976
40	8.0025	0073	0119	0165	0209	0253	0295	0336	0376	0415
50	8.0453	0491	0529	0565	0601	0636	0670	0703	0736	0768
60	8.0800	0831	0862	0892	0921	0950	0978	1006	1034	1062
70	8.1089	1116	1142	1167	1192	1217	1241	1265	1289	1312
80	8.1336	1359	1382	1404	1426	1448	1469	1490	1511	1532
90	8.1552	1572	1592	1612	1632	1651	1670	1689	1708	1727
100	8.1745	1763	1780	1796	1815	1833	1850	1867	1884	1901
110	8.1917	1934	1950	1966	1982	1998	2013	2029	2044	2059
120	8.2074	2089	2103	2118	2132	2147	2161	2175	2189	2203
130	8.2217	2231	2244	2258	2271	2284	2297	2310	2323	2336
140	8.2349	2361	2373	2386	2398	2410	2422	2435	2447	2459
150	8.2471	2483	2494	2506	2517	2529	2540	2552	2563	2575
160	8.2586	2597	2607	2618	2629	2640	2650	2661	2672	2683
170	8.2693	2703	2713	2723	2733	2744	2754	2764	2774	2784
180	8.2794	2804	2813	2823	2833	2843	2852	2862	2872	2882
190	8.2891	2900	2909	2918	2927	2937	2946	2955	2964	2973
200	8.2982	2991	3000	3009	3017	3026	3035	3044	3052	3061
210	8.3070	3079	3087	3096	3104	3112	3120	3129	3137	3146
220	8.3154	3162	3170	3178	3186	3194	3202	3210	3218	3226
230	8.3234	3242	3250	3258	3265	3273	3281	3289	3296	3304
240	8.3312	3320	3327	3335	3342	3350	3358	3366	3373	3381
250	8.3388	3396	3403	3410	3417	3425	3432	3439	3446	3454
260	8.3461	3468	3475	3482	3489	3496	3503	3510	3517	3524
270	8.3531	3538	3545	3552	3559	3566	3573	3580	3587	3594
280	8.3601	3608	3614	3621	3628	3635	3641	3648	3655	3662
290	8.3668	3675	3681	3688	3695	3702	3708	3715	3722	3729
300	8.3735	3742	3748	3755	3761	3768	3774	3781	3787	3794
310	8.3800	3807	3813	3820	3826	3832	3838	3845	3851	3858
320	8.3864	3871	3877	3884	3890	3896	3902	3909	3915	3922
330	8.3928	3934	3940	3946	3952	3959	3965	3971	3977	3983
340	8.3989	3995	4001	4008	4014	4020	4026	4033	4039	4045
350	8.4051	4057	4063	4070	4076	4082	4088	4095	4101	4107
360	8.4113	4119	4125	4131	4137	4144	4150	4156	4162	4168
370	8.4174	4180	4186	4192	4198	4204	4210	4216	4222	4228
380	8.4234	4240	4246	4252	4258	4264	4270	4276	4282	4288
390	8.4294	4300	4306	4312	4318	4324	4330	4336	4342	4348
400	8.4354	4360	4366	4372	4378	4385	4391	4397	4403	4409
410	8.4415	4421	4427	4433	4439	4445	4450	4456	4462	4468
420	8.4474	4480	4486	4492	4498	4504	4510	4516	4522	4528
430	8.4534	4540	4546	4552	4558	4564	4570	4576	4582	4588
440	8.4594	4600	4606	4612	4618	4624	4630	4636	4642	4648
450	8.4654	4660	4666	4673	4679	4685	4691	4698	4704	4710
460	8.4716	4722	4728	4734	4740	4746	4752	4758	4764	4770
470	8.4776	4782	4788	4794	4800	4806	4812	4818	4824	4830
480	8.4836	4842	4848	4855	4861	4867	4873	4880	4886	4892
490	8.4898	4904	4910	4916	4922	4929	4935	4941	4947	4953
500	8.4959	4965	4971	4978	4984	4990	4996	5003	5009	5015

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TABLE I.—G(v)—Continued.

$v^2/100$	0	1	2	3	4	5	6	7	8	9
510	8. 5021	5028	5034	5040	5046	5053	5059	5065	5071	5078
520	8. 5084	5091	5097	5103	5109	5116	5122	5128	5134	5141
530	8. 5147	5154	5160	5167	5173	5179	5185	5192	5198	5205
540	8. 5211	5218	5224	5231	5237	5243	5249	5256	5262	5269
550	8. 5275	5282	5288	5295	5301	5308	5314	5321	5327	5334
560	8. 5340	5347	5353	5360	5366	5373	5379	5386	5392	5399
570	8. 5405	5412	5418	5425	5432	5439	5445	5452	5459	5466
580	8. 5472	5479	5485	5492	5499	5506	5512	5519	5526	5533
590	8. 5539	5546	5553	5560	5566	5573	5580	5587	5593	5600
600	8. 5607	5614	5621	5628	5635	5642	5648	5655	5662	5669
610	8. 5676	5683	5690	5697	5704	5711	5717	5724	5731	5738
620	8. 5745	5752	5759	5766	5773	5781	5788	5795	5802	5809
630	8. 5816	5823	5830	5837	5844	5852	5859	5866	5873	5880
640	8. 5887	5894	5901	5909	5916	5923	5930	5938	5945	5952
650	8. 5959	5966	5973	5981	5988	5995	6002	6010	6017	6024
660	8. 6031	6039	6046	6054	6061	6068	6075	6083	6090	6098
670	8. 6105	6113	6120	6128	6135	6143	6150	6158	6165	6173
680	8. 6180	6188	6195	6203	6210	6218	6225	6233	6240	6248
690	8. 6255	6263	6270	6278	6286	6294	6301	6309	6317	6325
700	8. 6332	6340	6347	6355	6363	6371	6378	6386	6394	6402
710	8. 6409	6417	6425	6433	6441	6449	6456	6464	6472	6480
720	8. 6488	6496	6504	6512	6520	6528	6536	6544	6552	6560
730	8. 6568	6576	6584	6592	6600	6608	6616	6624	6632	6640
740	8. 6648	6656	6664	6672	6680	6689	6697	6705	6713	6721
750	8. 6729	6737	6745	6754	6762	6770	6778	6787	6795	6803
760	8. 6811	6820	6828	6837	6845	6853	6861	6870	6878	6887
770	8. 6895	6904	6912	6921	6929	6938	6946	6955	6963	6972
780	8. 6980	6989	6997	7006	7014	7023	7031	7040	7048	7057
790	8. 7065	7074	7082	7091	7099	7108	7117	7126	7134	7143
800	8. 7151	7160	7168	7177	7186	7195	7203	7212	7221	7230
810	8. 7238	7247	7256	7265	7274	7283	7292	7301	7310	7319
820	8. 7328	7337	7346	7354	7363	7372	7381	7390	7398	7407
830	8. 7416	7425	7434	7443	7452	7461	7470	7479	7488	7497
840	8. 7506	7515	7524	7533	7542	7552	7561	7570	7579	7588
850	8. 7597	7606	7615	7624	7633	7643	7652	7661	7670	7679
860	8. 7688	7698	7707	7716	7725	7735	7744	7753	7762	7772
870	8. 7781	7790	7799	7809	7818	7827	7836	7846	7855	7864
880	8. 7873	7883	7892	7902	7911	7920	7929	7939	7948	7958
890	8. 7967	7977	7986	7996	8005	8014	8023	8033	8042	8052
900	8. 8061	8071	8080	8090	8099	8108	8117	8127	8136	8146
910	8. 8155	8165	8174	8184	8193	8203	8213	8223	8232	8242
920	8. 8251	8261	8270	8280	8289	8299	8308	8318	8327	8337
930	8. 8346	8356	8365	8375	8384	8394	8404	8413	8423	8433
940	8. 8442	8452	8461	8471	8480	8490	8500	8510	8519	8529
950	8. 8538	8548	8557	8567	8576	8586	8595	8605	8614	8624
960	8. 8633	8643	8652	8662	8671	8681	8690	8700	8709	8719
970	8. 8728	8738	8747	8757	8766	8776	8785	8795	8804	8814
980	8. 8823	8833	8842	8852	8861	8871	8881	8891	8900	8910
990	8. 8919	8929	8938	8948	8957	8967	8976	8986	8995	9005
1000	8. 9014	9024	9033	9042	9051	9061	9070	9079	9088	9098

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TABLE I.—G(v)—Continued.

$v^2/100$	0	1	2	3	4	5	6	7	8	9
1010	8. 9107	9117	9126	9135	9144	9154	9163	9172	9181	9191
1020	8. 9200	9210	9219	9228	9237	9247	9256	9265	9274	9284
1030	8. 9293	9302	9311	9321	9330	9339	9348	9358	9367	9376
1040	8. 9385	9394	9403	9412	9421	9431	9440	9449	9458	9467
1050	8. 9476	9485	9494	9503	9512	9521	9530	9539	9548	9557
1060	8. 9566	9575	9584	9593	9601	9610	9619	9628	9636	9645
1070	8. 9654	9663	9671	9680	9689	9698	9706	9715	9724	9733
1080	8. 9741	9750	9758	9767	9775	9784	9792	9801	9809	9818
1090	8. 9826	9835	9843	9852	9860	9868	9876	9885	9893	9902
1100	8. 9910	9919	9927	9936	9944	9952	9960	9969	9977	9986
1110	8. 9994	*0002	*0010	*0018	*0026	*0035	*0043	*0051	*0059	*0067
1120	9. 0075	0083	0091	0099	0106	0114	0121	0129	0137	0145
1130	9. 0153	0161	0169	0177	0185	0193	0200	0208	0216	0224
1140	9. 0232	0240	0247	0255	0262	0270	0278	0286	0293	0301
1150	9. 0308	0316	0323	0331	0338	0345	0352	0360	0367	0375
1160	9. 0382	0389	0396	0404	0411	0418	0425	0433	0440	0447
1170	9. 0454	0461	0468	0475	0482	0489	0496	0503	0510	0517
1180	9. 0524	0531	0538	0545	0552	0559	0566	0573	0580	0587
1190	9. 0594	0601	0607	0614	0621	0628	0634	0641	0648	0655
1200	9. 0661	0668	0674	0681	0687	0694	0701	0708	0714	0721
1210	9. 0727	0734	0740	0747	0753	0759	0765	0772	0778	0785
1220	9. 0791	0797	0803	0809	0815	0822	0828	0834	0840	0846
1230	9. 0852	0858	0864	0870	0876	0882	0888	0894	0900	0906
1240	9. 0912	0918	0924	0930	0936	0942	0948	0954	0960	0966
1250	9. 0972	0978	0983	0989	0994	1000	1006	1012	1017	1023
1260	9. 1028	1034	1039	1045	1050	1056	1061	1067	1072	1078
1270	9. 1083	1089	1094	1100	1105	1110	1115	1121	1126	1132
1280	9. 1137	1142	1147	1153	1158	1163	1168	1174	1179	1184
1290	9. 1189	1194	1199	1204	1209	1215	1220	1225	1230	1235
1300	9. 1240	1245	1250	1255	1260	1265	1269	1274	1279	1284
1310	9. 1289	1294	1299	1304	1308	1313	1318	1323	1327	1332
1320	9. 1337	1342	1346	1351	1356	1361	1365	1370	1375	1380
1330	9. 1384	1389	1393	1398	1402	1407	1412	1417	1421	1426
1340	9. 1430	1435	1439	1444	1448	1452	1456	1461	1465	1470
1350	9. 1474	1479	1483	1487	1491	1496	1500	1504	1508	1513
1360	9. 1517	1521	1525	1530	1534	1538	1542	1547	1551	1555
1370	9. 1559	1563	1567	1571	1575	1579	1583	1587	1591	1595
1380	9. 1599	1603	1607	1611	1615	1619	1623	1627	1631	1635
1390	9. 1639	1643	1647	1651	1655	1659	1662	1666	1670	1674
1400	9. 1678	1682	1685	1689	1693	1697	1700	1704	1708	1712
1410	9. 1715	1719	1722	1726	1730	1734	1737	1741	1745	1749
1420	9. 1752	1756	1759	1763	1766	1770	1774	1778	1781	1785
1430	9. 1788	1792	1795	1799	1802	1805	1808	1812	1815	1819
1440	9. 1822	1826	1829	1833	1836	1840	1843	1847	1850	1854
1450	9. 1857	1861	1864	1867	1870	1874	1877	1880	1883	1887
1460	9. 1890	1893	1896	1900	1903	1906	1909	1913	1916	1919
1470	9. 1922	1925	1928	1931	1934	1938	1941	1944	1947	1950
1480	9. 1953	1956	1959	1962	1965	1969	1972	1975	1978	1981
1490	9. 1984	1987	1990	1993	1996	1999	2002	2005	2008	2011
1500	9. 2014	2017	2020	2023	2026	2029	2032	2035	2038	2041

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TABLE I.—G(v)—Continued.

$v^2/100$	0	1	2	3	4	5	6	7	8	9
1510	9. 2044	2047	2050	2053	2055	2058	2061	2064	2066	2069
1520	9. 2072	2075	2078	2081	2083	2086	2089	2092	2094	2097
1530	9. 2100	2103	2106	2109	2111	2114	2117	2120	2122	2125
1540	9. 2128	2131	2133	2136	2139	2142	2144	2147	2150	2153
1550	9. 2155	2158	2160	2163	2166	2169	2171	2174	2177	2180
1560	9. 2182	2185	2187	2190	2192	2195	2197	2200	2202	2205
1570	9. 2207	2210	2212	2215	2217	2220	2222	2225	2227	2230
1580	9. 2232	2235	2237	2240	2242	2245	2247	2250	2252	2255
1590	9. 2257	2260	2262	2265	2267	2270	2272	2275	2277	2280
1600	9. 2282	2285	2287	2289	2291	2294	2296	2299	2301	2304
1610	9. 2306	2308	2310	2313	2315	2318	2320	2322	2324	2327
1620	9. 2329	2331	2333	2336	2338	2340	2342	2345	2347	2349
1630	9. 2351	2353	2355	2358	2360	2362	2364	2367	2369	2371
1640	9. 2373	2375	2377	2380	2382	2384	2386	2389	2391	2393
1650	9. 2395	2397	2399	2402	2404	2406	2408	2411	2413	2415
1660	9. 2417	2419	2421	2423	2425	2428	2430	2432	2434	2436
1670	9. 2438	2440	2442	2444	2446	2449	2451	2453	2455	2457
1680	9. 2459	2461	2463	2465	2467	2469	2471	2473	2475	2477
1690	9. 2479	2481	2483	2485	2487	2489	2491	2493	2495	2497
1700	9. 2499	2501	2503	2505	2507	2509	2511	2513	2515	2517
1710	9. 2519	2521	2523	2525	2527	2529	2531	2533	2535	2537
1720	9. 2539	2541	2543	2545	2546	2548	2550	2552	2554	2556
1730	9. 2558	2560	2561	2563	2565	2567	2569	2571	2572	2574
1740	9. 2576	2578	2580	2582	2583	2585	2587	2589	2591	2593
1750	9. 2595	2597	2598	2600	2602	2604	2606	2608	2609	2611
1760	9. 2613	2615	2617	2619	2620	2622	2624	2626	2627	2629
1770	9. 2631	2633	2634	2636	2638	2640	2641	2643	2645	2647
1780	9. 2648	2650	2651	2653	2655	2657	2658	2660	2662	2664
1790	9. 2665	2667	2668	2670	2672	2674	2675	2677	2679	2681
1800	9. 2682	2684	2685	2687	2689	2691	2692	2694	2695	2697
1810	9. 2699	2701	2702	2704	2705	2707	2708	2710	2712	2714
1820	9. 2715	2717	2718	2720	2721	2723	2725	2727	2728	2730
1830	9. 2731	2732	2734	2736	2737	2739	2741	2743	2744	2746
1840	9. 2747	2749	2750	2752	2753	2755	2757	2758	2760	2761
1850	9. 2763	2765	2766	2768	2769	2771	2773	2774	2776	2777
1860	9. 2779	2781	2782	2784	2785	2787	2788	2790	2791	2793
1870	9. 2794	2796	2797	2799	2800	2802	2803	2805	2806	2808
1880	9. 2809	2810	2812	2813	2815	2816	2818	2819	2821	2822
1890	9. 2824	2825	2826	2828	2829	2831	2832	2834	2835	2837
1900	9. 2838	2839	2841	2842	2844	2845	2847	2848	2850	2851
1910	9. 2853	2854	2855	2857	2858	2860	2861	2863	2864	2866
1920	9. 2867	2868	2870	2871	2873	2874	2875	2877	2878	2880
1930	9. 2881	2882	2884	2885	2887	2888	2889	2891	2892	2894
1940	9. 2895	2896	2898	2899	2900	2902	2903	2904	2906	2907
1950	9. 2909	2910	2911	2913	2914	2915	2917	2918	2919	2921
1960	9. 2922	2923	2925	2926	2927	2929	2930	2931	2932	2934
1970	9. 2935	2936	2938	2939	2940	2942	2943	2944	2945	2947
1980	9. 2948	2949	2951	2952	2953	2955	2956	2957	2958	2960
1990	9. 2961	2962	2964	2965	2966	2968	2969	2970	2971	2973
2000	9. 2974	2975	2976	2978	2979	2980	2981	2983	2984	2985

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TABLE I.—G(v)—Continued.

$v^2/100$	00	10	20	30	40	50	60	70	80	90
2000	9. 2974	2986	2999	3011	3023	3035	3047	3058	3070	3082
2100	9. 3093	3105	3116	3127	3137	3148	3158	3168	3179	3189
2200	9. 3199	3209	3219	3228	3238	3248	3258	3267	3277	3286
2300	9. 3295	3304	3313	3322	3331	3339	3348	3356	3365	3373
2400	9. 3381	3389	3397	3405	3413	3421	3429	3436	3444	3452
2500	9. 3459	3467	3474	3481	3489	3496	3503	3510	3517	3524
2600	9. 3531	3538	3545	3551	3558	3565	3572	3578	3585	3591
2700	9. 3598	3604	3610	3616	3623	3629	3635	3641	3647	3653
2800	9. 3659	3665	3671	3676	3682	3688	3693	3699	3704	3710
2900	9. 3715	3721	3726	3731	3737	3742	3747	3753	3758	3764
3000	9. 3769	3774	3779	3783	3788	3793	3799	3804	3809	3814
3100	9. 3819	3823	3828	3833	3837	3842	3846	3851	3856	3860
3200	9. 3865	3870	3874	3879	3883	3888	3892	3897	3901	3906
3300	9. 3910	3914	3918	3922	3926	3931	3935	3939	3943	3947
3400	9. 3951	3955	3959	3963	3967	3971	3975	3979	3983	3987
3500	9. 3991	3995	3999	4002	4006	4010	4014	4018	4021	4025
3600	9. 4029	4033	4036	4040	4043	4047	4051	4054	4058	4061
3700	9. 4065	4069	4072	4076	4079	4083	4086	4090	4093	4097
3800	9. 4100	4103	4107	4110	4113	4117	4120	4123	4126	4130
3900	9. 4133	4136	4139	4143	4146	4149	4152	4155	4159	4162
4000	9. 4165	4168	4171	4174	4177	4181	4184	4187	4190	4193
4100	9. 4196	4199	4202	4205	4208	4211	4214	4217	4220	4223
4200	9. 4226	4229	4232	4234	4237	4240	4243	4246	4249	4251
4300	9. 4254	4257	4260	4262	4265	4268	4271	4274	4276	4279
4400	9. 4282	4285	4287	4290	4293	4295	4298	4301	4303	4306
4500	9. 4309	4311	4314	4317	4319	4322	4324	4327	4330	4332
4600	9. 4335	4338	4340	4343	4345	4348	4350	4353	4355	4358
4700	9. 4360	4363	4365	4368	4370	4373	4375	4378	4380	4383
4800	9. 4385	4387	4390	4392	4395	4397	4400	4402	4405	4407
4900	9. 4410	4412	4415	4417	4419	4422	4424	4427	4429	4432
5000	9. 4434	4436	4439	4441	4443	4446	4448	4451	4453	4455
5100	9. 4458	4460	4462	4465	4467	4469	4472	4474	4476	4479
5200	9. 4481	4483	4486	4488	4490	4492	4495	4497	4499	4501
5300	9. 4504	4506	4508	4510	4513	4515	4517	4519	4522	4524
5400	9. 4526	4528	4530	4533	4535	4537	4539	4541	4544	4546
5500	9. 4548	4550	4552	4555	4557	4559	4561	4563	4566	4568
5600	9. 4570	4572	4574	4576	4579	4581	4583	4585	4587	4589
5700	9. 4592	4594	4596	4598	4600	4602	4604	4607	4609	4611
5800	9. 4613	4615	4617	4619	4621	4624	4626	4628	4630	4632
5900	9. 4634	4636	4638	4640	4642	4645	4647	4649	4651	4653
6000	9. 4655	4657	4659	4661	5663	4665	4667	4669	4671	4674
6100	9. 4676	4678	4680	4682	4684	4686	4688	4690	4692	4694
6200	9. 4696	4698	4700	4702	4704	4706	4708	4710	4712	4714
6300	9. 4716	4718	4720	4722	4724	4726	4728	4730	4732	4734
6400	9. 4736	4738	4740	4742	4744	4746	4748	4750	4752	4754
6500	9. 4756	4758	4760	4762	4764	4766	4768	4770	4772	4774
6600	9. 4776	4778	4780	4782	4784	4786	4788	4790	4792	4794
6700	9. 4796	4797	4799	4801	4803	4805	4807	4809	4811	4813
6800	9. 4815	4817	4819	4821	4823	4825	4827	4829	4831	4833
6900	9. 4835	4836	4838	4840	4842	4844	4846	4848	4850	4852
7000	9. 4854	4856	4858	4860	4862	4864	4866	4868	4870	4872

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TABLE I.—G(v)—Continued.

$v^2/100$	00	10	20	30	40	50	60	70	80	90
7100	9. 4874	4875	4877	4879	4881	4883	4885	4887	4889	4891
7200	9. 4893	4895	4897	4899	4901	4903	4904	4906	4908	4910
7300	9. 4912	4914	4916	4918	4920	4922	4923	4925	4927	4929
7400	9. 4931	4933	4935	4937	4939	4941	4942	4944	4946	4948
7500	9. 4950	4952	4954	4956	4958	4960	4961	4963	4965	4967
7600	9. 4969	4971	4973	4975	4976	4978	4980	4982	4984	4986
7700	9. 4988	4989	4991	4993	4995	4997	4999	5000	5002	5004
7800	9. 5006	5008	5010	5012	5013	5015	5017	5019	5021	5023
7900	9. 5025	5026	5028	5030	5032	5034	5036	5037	5039	5041
8000	9. 5043	5045	5047	5049	5050	5052	5054	5056	5058	5060
$v^2/100$	000	100	200	300	400	500	600	700	800	900
8000	9. 5043	5062	5080	5098	5117	5135	5153	5171	5189	5206
9000	9. 5225	5242	5259	5277	5294	5312	5329	5347	5364	5381
10000	9. 5399	5416	5433	5450	5467	5484	5501	5518	5535	5551
11000	9. 5568	5585	5601	5618	5634	5650	5667	5683	5699	5715
12000	9. 5731	5747	5763	5779	5795	5811	5826	5842	5857	5873
13000	9. 5888	5903	5918	5933	5948	5963	5977	5991	6006	6020
14000	9. 6034	6048	6062	6075	6089	6103	6117	6131	6144	6158
15000	9. 6172	6185	6199	6212	6226	6239	6252	6265	6278	6291
16000	9. 6304	6317	6329	6342	6354	6367	6379	6392	6404	6417
17000	9. 6429	6441	6453	6465	6477	6489	6501	6513	6525	6537
18000	9. 6549	6560	6572	6583	6595	6606	6617	6628	6640	6651
19000	9. 6662	6673	6684	6694	6705	6716	6727	6737	6748	6758
20000	9. 6769	6780	6790	6801	6811	6822	6832	6842	6853	6863
21000	9. 6873	6883	6893	6903	6913	6923	6933	6943	6953	6963
22000	9. 6973	6983	6992	7002	7011	7021	7030	7040	7049	7059
23000	9. 7068	7077	7086	7096	7105	7114	7123	7132	7141	7150
24000	9. 7159	7168	7177	7185	7194	7203	7212	7220	7229	7237
25000	9. 7246	7255	7263	7272	7280	7289	7297	7306	7314	7323
26000	9. 7331	7339	7347	7355	7363	7371	7379	7387	7396	7404
27000	9. 7412	7420	7428	7436	7444	7452	7460	7467	7475	7482
28000	9. 7490	7498	7505	7513	7520	7528	7536	7543	7551	7558
29000	9. 7566	7573	7581	7588	7596	7603	7610	7617	7625	7632
30000	9. 7639	7646	7653	7661	7668	7675	7682	7689	7696	7703
31000	9. 7710	7717	7724	7731	7738	7745	7752	7759	7765	7772
32000	9. 7779	7786	7793	7799	7806	7813	7820	7827	7833	7840

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TABLE II.—H(y).

The table gives $\log_{10} H$, the argument being y, expressed in meters. The function H is defined by the formula

$$H = 10^{-.000045y},$$

and the five-place logarithms given in the table are exact. The characteristic $9(-10)$ is to be supplied throughout, except for $y=0$; e. g., $\log H(y=1430) = 9.93565 - 10$. The table can also be used for values of y greater than 10,000, by supposing the argument multiplied by 10, and regarding the first tabulated figure in the logarithm as characteristic; e. g., $\log H(y=14300) = 9.3565$.

y	0	10	20	30	40	50	60	70	80	90
0	00000	99955	99910	99865	99820	99775	99730	99685	99640	99595
100	99550	99505	99460	99415	99370	99325	99280	99235	99190	99145
200	99100	99055	99010	98965	98920	98875	98830	98785	98740	98695
300	98650	98605	98560	98515	98470	98425	98380	98335	98290	98245
400	98200	98155	98110	98065	98020	97975	97930	97885	97840	97795
500	97750	97705	97660	97615	97570	97525	97480	97435	97390	97345
600	97300	97255	97210	97165	97120	97075	97030	96985	96940	96895
700	96850	96805	96760	96715	96670	96625	96580	96535	96490	96445
800	96400	96355	96310	96265	96220	96175	96130	96085	96040	95995
900	95950	95905	95860	95815	95770	95725	95680	95635	95590	95545
1000	95500	95455	95410	95365	95320	95275	95230	95185	95140	95095
1100	95050	95005	94960	94915	94870	94825	94780	94735	94690	94645
1200	94600	94555	94510	94465	94420	94375	94330	94285	94240	94195
1300	94150	94105	94060	94015	93970	93925	93880	93835	93790	93745
1400	93700	93655	93610	93565	93520	93475	93430	93385	93340	93295
1500	93250	93205	93160	93115	93070	93025	92980	92935	92890	92845
1600	92800	92755	92710	92665	92620	92575	92530	92485	92440	92395
1700	92350	92305	92260	92215	92170	92125	92080	92035	91990	91945
1800	91900	91855	91810	91765	91720	91675	91630	91585	91540	91495
1900	91450	91405	91360	91315	91270	91225	91180	91135	91090	91045
2000	91000	90955	90910	90865	90820	90775	90730	90685	90640	90595
2100	90550	90505	90460	90415	90370	90325	90280	90235	90190	90145
2200	90100	90055	90010	89965	89920	89875	89830	89785	89740	89695
2300	89650	89605	89560	89515	89470	89425	89380	89335	89290	89245
2400	89200	89155	89110	89065	89020	88975	88930	88885	88840	88795
2500	88750	88705	88660	88615	88570	88525	88480	88435	88390	88345
2600	88300	88255	88210	88165	88120	88075	88030	87985	87940	87895
2700	87850	87805	87760	87715	87670	87625	87580	87535	87490	87445
2800	87400	87355	87310	87265	87220	87175	87130	87085	87040	86995
2900	86950	86905	86860	86815	86770	86725	86680	86635	86590	86545
3000	86500	86455	86410	86365	86320	86275	86230	86185	86140	86095
3100	86050	86005	85960	85915	85870	85825	85780	85735	85690	85645
3200	85600	85555	85510	85465	85420	85375	85330	85285	85240	85195
3300	85150	85105	85060	85015	84970	84925	84880	84835	84790	84745
3400	84700	84655	84610	84565	84520	84475	84430	84385	84340	84295
3500	84250	84205	84160	84115	84070	84025	83980	83935	83890	83845
3600	83800	83755	83710	83665	83620	83575	83530	83485	83440	83395
2700	83350	83305	83260	83215	83170	83125	83080	83035	82990	82945
3800	82900	82855	82810	82765	82720	82675	82630	82585	82540	82495
3900	82450	82405	82360	82315	82270	82225	82180	82135	82090	82045
4000	82000	81955	81910	81865	81820	81775	81730	81685	81640	81595
4100	81550	81505	81460	81415	81370	81325	81280	81235	81190	81145
4200	81100	81055	81010	80965	80920	80875	80830	80785	80740	80695
4300	80650	80605	80560	80515	80470	80425	80380	80335	80290	80245
4400	80200	80155	80110	80065	80020	79975	79930	79885	79840	79795
4500	79750	79705	79660	79615	79570	79525	79480	79435	79390	79345
4600	79300	79255	79210	79165	79120	79075	79030	78985	78940	78895
4700	78850	78805	78760	78715	78670	78625	78580	78535	78490	78445
4800	78400	78355	78310	78265	78220	78175	78130	78085	78040	77995
4900	77950	77905	77860	77815	77770	77725	77680	77635	77590	77545
5000	77500	77455	77410	77365	77320	77275	77230	77185	77140	77095

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TABLE II.—H(y)—Continued.

y	0	10	20	30	40	50	60	70	80	90
5100	77050	77005	76960	76915	76870	76825	76780	76735	76690	76645
5200	76600	76555	76510	76465	76420	76375	76330	76285	76240	76195
5300	76150	76105	76060	76015	75970	75925	75880	75835	75790	75745
5400	75700	75655	75610	75565	75520	75475	75430	75385	75340	75295
5500	75250	75205	75160	75115	75070	75025	74980	74935	74890	74845
5600	74800	74755	74710	74665	74620	74575	74530	74485	74440	74395
5700	74350	74305	74260	74215	74170	74125	74080	74035	73990	73945
5800	73900	73855	73810	73765	73720	73675	73630	73585	73540	73495
5900	73450	73405	73360	73315	73270	73225	73180	73135	73090	73045
6000	73000	72955	72910	72865	72820	72775	72730	72685	72640	72595
6100	72550	72505	72460	72415	72370	72325	72280	72235	72190	72145
6200	72100	72055	72010	71965	71920	71875	71830	71785	71740	71695
6300	71650	71605	71560	71515	71470	71425	71380	71335	71290	71245
6400	71200	71155	71110	71065	71020	70975	70930	70885	70840	70795
6500	70750	70705	70660	70615	70570	70525	70480	70435	70390	70345
6600	70300	70255	70210	70165	70120	70075	70030	69985	69940	69895
6700	69850	69805	69760	69715	69670	69625	69580	69535	69490	69445
6800	69400	69355	69310	69265	69220	69175	69130	69085	69040	68995
6900	68950	68905	68860	68815	68770	68725	68680	68635	68590	68545
7000	68500	68455	68410	68365	68320	68275	68230	68185	68140	68095
7100	68050	68005	67960	67915	67870	67825	67780	67735	67690	67645
7200	67600	67555	67510	67465	67420	67375	67330	67285	67240	67195
7300	67150	67105	67060	67015	66970	66925	66880	66835	66790	66745
7400	66700	66655	66610	66565	66520	66475	66430	66385	66340	66295
7500	66250	66205	66160	66115	66070	66025	65980	65935	65890	65845
7600	65800	65755	65710	65665	65620	65575	65530	65485	65440	65395
7700	65350	65305	65260	65215	65170	65125	65080	65035	64990	64945
7800	64900	64855	64810	64765	64720	64675	64630	64585	64540	64495
7900	64450	64405	64360	64315	64270	64225	64180	64135	64090	64045
8000	64000	63955	63910	63865	63820	63775	63730	63685	63640	63595
8100	63550	63505	63460	63415	63370	63325	63280	63235	63190	63145
8200	63100	63055	63010	62965	62920	62875	62830	62785	62740	62695
8300	62650	62605	62560	62515	62470	62425	62380	62335	62290	62245
8400	62200	62155	62110	62065	62020	61975	61930	61885	61840	61795
8500	61750	61705	61660	61615	61570	61525	61480	61435	61390	61345
8600	61300	61255	61210	61165	61120	61075	61030	60985	60940	60895
8700	60850	60805	60760	60715	60670	60625	60580	60535	60490	60445
8800	60400	60355	60310	60265	60220	60175	60130	60085	60040	59995
8900	59950	59905	59860	59815	59770	59725	59680	59635	59590	59545
9000	59500	59455	59410	59365	59320	59275	59230	59185	59140	59095
9100	59050	59005	58960	58915	58870	58825	58780	58735	58690	58645
9200	58600	58555	58510	58465	58420	58375	58330	58285	58240	58195
9300	58150	58105	58060	58015	57970	57925	57880	57835	57790	57745
9400	57700	57655	57610	57565	57520	57475	57430	57385	57340	57295
9500	57250	57205	57160	57115	57070	57025	56980	56935	56890	56845
9600	56800	56755	56710	56665	56620	56575	56530	56485	56440	56395
9700	56350	56305	56260	56215	56170	56125	56080	56035	55990	55945
9800	55900	55855	55810	55765	55720	55675	55630	55585	55540	55495
9900	55450	55405	55360	55315	55270	55225	55180	55135	55090	55045
10000	55000	54955	54910	54865	54820	54775	54730	54685	54640	54595

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TABLE III.—$d \log G/(vdv)$.

The table gives the numerical value of $d \log_e G/(vdv)$, the argument being $v^2/100$, where v is expressed in meters per second. The expression $.0_3463$ means .000463, and $.0_5774$ means .00000774, etc.

Since $H = 10^{-.000045y} = e^{-.0001036y}$, the expression $d \log_e H/dy$ has the constant value $-.0001036$.

$v^2/100$	00	20	40	60	80	100	120	140	160	180
00	.....	$.0_3463$	223	134	106	*835	*688	*585	*511	*454
200	$.0_4412$	378	352	332	315	303	293	286	281	278
400	$.0_4276$	276	277	279	282	285	289	295	300	307
600	$.0_4314$	322	329	338	346	355	364	373	382	391
800	$.0_4400$	409	416	424	429	434	438	440	440	438
1000	$.0_4434$	428	420	410	398	384	370	354	338	322
1200	$.0_4305$	289	274	258	244	231	218	206	195	185
1400	$.0_4175$	167	158	151	144	137	131	125	120	115
1600	$.0_4110$	106	103	*984	*948	*915	*884	*854	*826	*800
1800	$.0_5774$	751	729	708	687	668	650	632	616	600
2000	$.0_5585$	570	556	542	530	517	506	494	483	473
2200	$.0_5462$	453	443	434	425	417	408	400	393	385
2400	$.0_5378$	371	364	358	352	345	339	334	328	323
2600	$.0_5317$	312	307	302	297	293	288	284	280	276
2800	$.0_5271$	268	264	260	256	253	249	246	243	240
3000	$.0_5237$	234	231	228	225	222	220	217	214	212
3200	$.0_5209$	207	205	202	200	198	196	194	192	190
3400	$.0_5188$	186	184	182	180	179	177	175	174	172
3600	$.0_5171$	169	168	166	165	163	162	161	159	158
3800	$.0_5157$	155	154	153	152	151	149	148	147	146
4000	$.0_5145$	144	143	142	141	140	139	138	138	137
4200	$.0_5136$	136	135	134	133	132	132	131	130	129
4400	$.0_5128$	128	127	126	126	125	124	124	123	122
4600	$.0_5121$	120	120	119	119	118	118	117	116	115
4800	$.0_5114$	114	113	112	112	111	110	110	109	109
5000	$.0_5108$	108	107	107	106	106	106	105	105	105
5200	$.0_5104$	104	104	104	103	103	103	103	102	102
5400	$.0_5102$	102	101	101	101	101	100	100	100	*998
5600	$.0_6996$	993	991	989	987	985	983	981	979	977
5800	$.0_6975$	973	971	969	967	965	963	961	959	957
6000	$.0_6955$	953	951	949	947	946	944	942	940	938
6200	$.0_6937$	935	934	932	930	929	927	926	924	922
6400	$.0_6921$	919	918	916	915	914	912	911	910	909
6600	$.0_6908$	907	906	905	904	903	901	900	899	898
6800	$.0_6897$	896	894	893	892	891	890	889	888	887
7000	$.0_6886$	884	883	882	881	880	879	878	877	876
7200	$.0_6875$	874	873	872	871	870	869	868	867	866
7400	$.0_6865$	864	863	862	861	860	859	858	857	856
7600	$.0_6855$	854	853	852	851	850	849	848	847	846
7800	$.0_6845$	844	844	843	842	841	840	839	838	837
8000	$.0_6836$	835	834	833	832	831	830	830	829	828

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TABLE III.—*d* log G/(*vdv*)—Continued.

$v^2/100$	00	20	40	60	80	100	120	140	160	180
8200	.$0_6$827	826	825	824	823	822	822	821	820	819
8400	.$0_6$818	818	817	816	815	814	814	813	812	811
8600	.$0_6$810	810	809	808	807	806	805	805	804	803
8800	.$0_6$802	801	800	800	799	798	797	796	795	795
9000	.$0_6$794	793	792	791	790	790	789	788	787	787
9200	.$0_6$786	785	785	784	783	782	782	781	780	780
9400	.$0_6$779	778	778	777	776	775	775	774	773	772
9600	.$0_6$771	771	770	769	768	768	767	766	766	765
9800	.$0_6$764	763	762	762	761	760	760	759	758	758
$v^2/100$	000	200	400	600	800	1000	1200	1400	1600	1800
10000	.$0_6$757	749	742	736	729	723	717	711	705	699
12000	.$0_6$693	687	681	675	669	664	658	653	648	642
14000	.$0_6$637	632	626	621	617	611	606	601	595	590
16000	.$0_6$586	580	575	571	565	560	556	550	545	539
18000	.$0_6$534	530	526	521	515	510	506	502	497	492
20000	.$0_6$488	483	479	475	470	466	462	458	454	450
22000	.$0_6$446	442	438	434	430	425	422	419	417	414
24000	.$0_6$411	408	405	402	399	396	393	390	386	383
26000	.$0_6$380	378	375	373	370	368	365	362	360	357
28000	.$0_6$354	352	350	347	345	343	341	339	336	334
30000	.$0_6$332	330	328	325	323	321	319	317	315	313
32000	.$0_6$311	309	307	305	303	301	.....	.....	.....	.....

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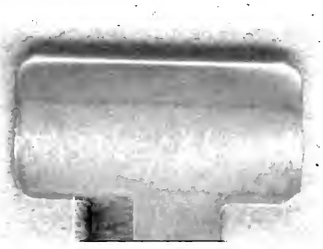

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